Forbidden graph minors, Arkhipov's theorem, and linear system games

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Non-local games

Given a simple connected graph $G=(V,E)$ with a non-proper vertex 2-colouring $b: V \to \{0,1\}$ let $E(v) \subseteq E$ be the edges incident to $v \in V$.

To the pair (G,b) there is an associated two-player non-local game $\mathcal{G}$.

Each player receives a vertex u, v, and responds with $f, g : E \to \{0,1\}$ such that $\sum_{e \in E(u)} f(e) = b(u)$ and $\sum_{e \in E(v)} g(e) = b(v)$.

They win if $f(e) = g(e)$ for every $e \in E(u) \cap E(v)$.

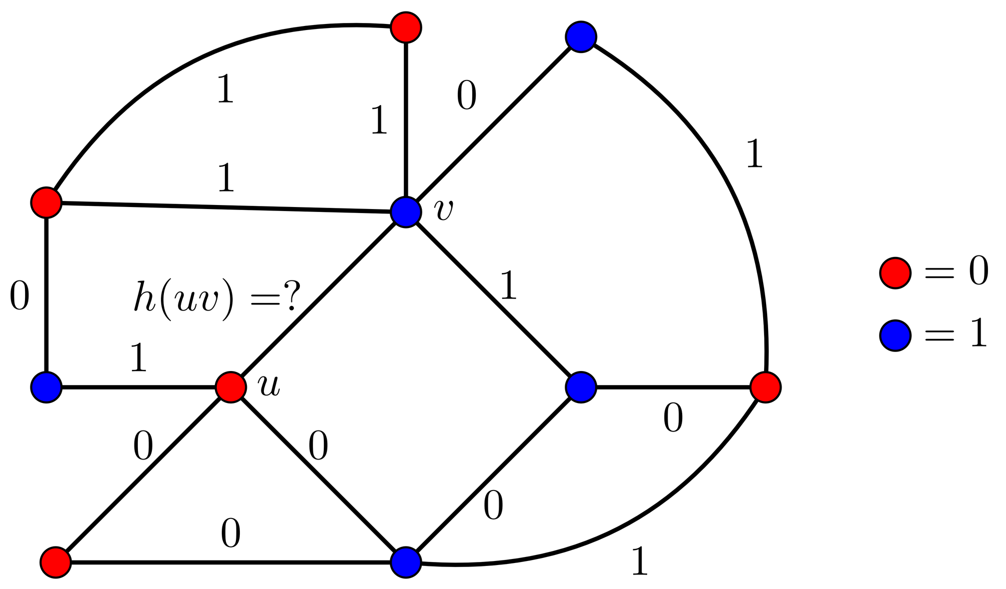

Figure 1: A game with no perfect strategy.

A *deterministic strategy* for $\mathcal{G}$ is a function $h: E \to \{0,1\}$, such that $\sum_{e \in E(v)} h(e) = b(v)$ for all $v \in V$.

This non-local game $\mathcal{G}$ is a linear system game for the linear system $Ax = b$ over $\mathbb{Z}_2$, where A is the *incidence matrix* of G.

Solution groups

The *solution group* $\Gamma(G,b)$ is the finitely-presented group with generators $\{x_e : e \in E\} \cup \{J\}$ and relations:

(a) $x_e^2 = 1$ for all $e \in E$,

(b) $\prod_{e \in E(v)} x_e = J^{b(v)}$ for all $v \in V$,

(c) $[x_e, x_{e'}] = 1$ for all $v \in V$ and pairs $e, e' \in E(v)$,

(d) $J^2 = 1$ and $[x_e, J] = 1$ for all $e \in E$.

A linear system game over $\mathbb{Z}_2$ has a perfect quantum strategy iff there is a representation $\pi : \Gamma(G,b) \to \mathcal{U}(\mathcal{H})$ such that $\pi(J) \neq \mathbb{1}_{\mathcal{H}}$.

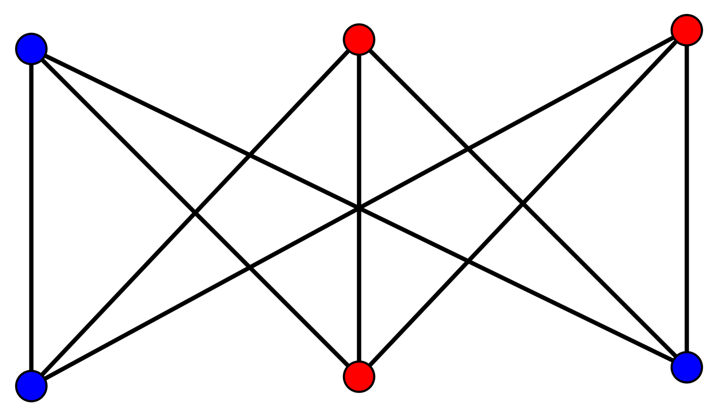

Figure 2: $\mathcal{G}(K_{3,3}, b)$ with b odd parity is the Mermin-Peres magic square game, which has a perfect quantum strategy but no perfect classical strategy.

Graph minor operations

The standard *graph minor* operations are edge deletion, vertex deletion, and edge contraction.

On 2-coloured graphs (G,b), the edge contraction minor combines the parity of two endpoints.

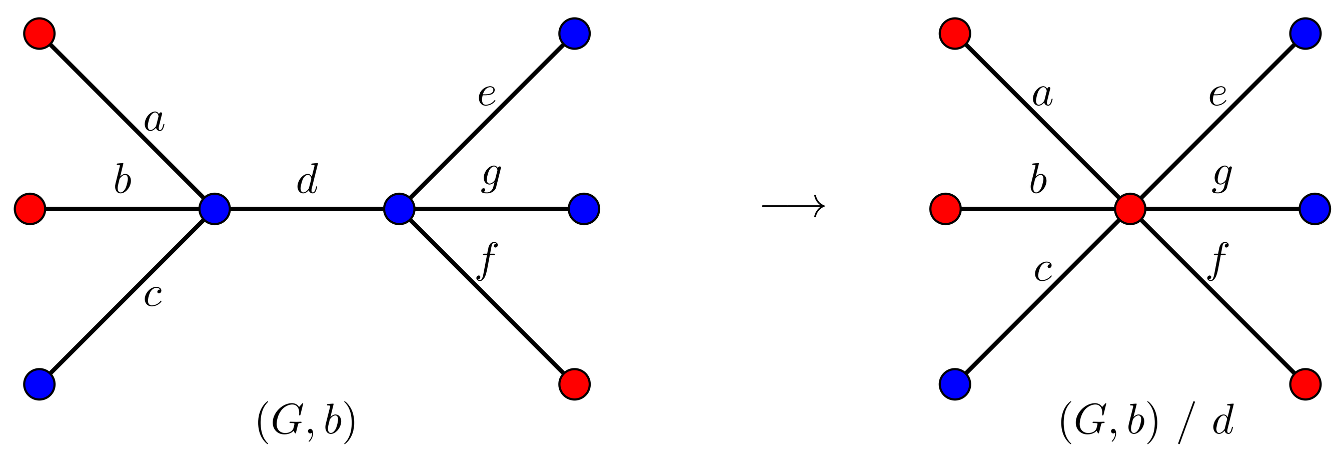

Figure 3: Two blue vertices become one red vertex under the edge contraction of d.

We add an additional minor operation to graphs (G,b), swapping the colours at the ends of an edge, and disallow deletion of blue vertices.

A graph (G,b) contains (H,b') as a minor if there exists a sequence of graph minor operations that take $(G,b) \to (H,b')$.

Wagner's theorem. A graph is *planar* iff it does not contain $K_{3,3}$ or K_5 as minors.

Robertson-Seymour theorem. For every graph minor closed property P, there is a a finite list of graphs $\mathcal{L}$ such that a graph satisfies P iff it does not contain any minors in $\mathcal{L}$.

Deciding whether G contains any minors in a finite list $\mathcal{L}$ can be done in time $O(V^3)$.

Main question and results

Arkhipov's theorem. If b has odd parity, then $\mathcal{G}(G,b)$ has a perfect quantum strategy if and only if G is non-planar.

Question: Can we explain the connection in Arkhipov's theorem between solution group properties and graph minors?

Answer: Yes! We can even prove analogous results:

Theorem 1. If b has even parity, then $\Gamma(G,b)$ is an *abelian* group if and only if (G,b) does not contain $K_{3,4}$ or two vertex-disjoint cycles $C \sqcup C$ as minors.

Theorem 2. $\Gamma(G,b)$ is a *finite* group if and only if it does not contain $K_{3,6}$ or two vertex-disjoint cycles $C \sqcup C$ as minors.

Solution groups and graph minors

Lemma. If (H,b') is a minor of (G,b) then there is a *surjective group homomorphism* $\phi : \Gamma(G,b) \twoheadrightarrow \Gamma(H,b')$.

Example. The minor $G \to C \sqcup C$ corresponds to $\Gamma(G,b) \twoheadrightarrow \mathbb{Z}_2 * \mathbb{Z}_2$.

Corollary. Every *quotient-closed* property of solution groups $\Gamma(G,b)$ is characterized by a finite set of forbidden minors.

Proofs of Theorems 1 and 2: use classification of graphs which do not contain two vertex-disjoint cycles (Lovász, 1965).

Solution groups and quantum strategies

Abelianness (see Theorem 1)

$\Gamma(G,b)$ abelian $\Leftrightarrow$ all irreducible perfect strategies are 1-dimensional (1-dimensional = deterministic $\subset$ classical).

If b has even parity then the game $\mathcal{G}$ has a perfect deterministic strategy.

If $\Gamma(G,b)$ is non-abelian and b has even parity, then there are both deterministic and higher-dimensional irreducible perfect strategies.

Finiteness (see Theorem 2)

$\Gamma(G,b)$ finite $\Rightarrow$ all irreducible perfect strategies are finite dimensional.

Open problems

The property that all perfect strategies of $\mathcal{G}(G,b)$ have classical correlation matrices is a quotient-closed property of $\Gamma(G,b)$.

Can we find the minors for this property?

Conjecture: All perfect strategies of $\mathcal{G}(G,b)$ have classical correlation matrices if and only if b has even parity.

What are the forbidden graph minors for other quotient-closed properties of the solution group (e.g. *amenability*)?

What other group theory features of $\Gamma(G,b)$ can be related to graph theory features of (G,b)?

References

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