The pretty bad measurement and optimal bounds for antidistinguishability

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Quantum State Discrimination

Helen selects ρ_i from $\{\rho_1, \rho_2, \ldots, \rho_k\}$ with probability p_i and sends it to Bob. Bob's task is to figure out the index i.

$$\rho_i \longrightarrow$$

$$\longleftarrow i$$

The best Bob can do, denoted by $\mathcal{P}_{\text{best}}$, is given by

$$\mathcal{P}_{\text{best}} = \max \sum_{i=1}^{k} p_i \langle M_i, \rho_i \rangle$$

Motivating thought experiment

What is the worst Bob can do? In other words, how well can Bob do at the antigame?

$$\mathcal{P}_{\text{worst}} = \min \sum_{i=1}^{k} p_i \langle M_i, \rho_i \rangle$$

Pretty Good Measurement

Define $P = \sum_{i=1}^{k} p_i \rho_i$. The pretty good measurement (PGM) operators are defined as

$$G_i := P^{-1/2} (p_i \rho_i) P^{-1/2}$$

Why do we care?

Barnum and Knill [2000] showed that $\mathcal{P}_{\text{PGM}} = \sum_{i=1}^{k} p_i \langle G_i, \rho_i \rangle$ approximates $\mathcal{P}_{\text{best}}$ by

$$\mathcal{P}_{\text{best}}^2 \leq \mathcal{P}_{\text{PGM}} \leq \mathcal{P}_{\text{best}}$$

Question

Does $\mathcal{P}_{\text{worst}}$ have a corresponding measurement that approximates it?

What is the Pretty Bad Measurement?

Pretty bad measurement (PBM) operators are given by

$$B_i := P^{-1/2} \left(\frac{1}{k-1} \sum_{j \neq i} p_j \rho_j \right) P^{-1/2}$$

Perform the PGM and randomly pick a different state.

Relationships

$$\mathcal{P}_{\text{PBM}} = \frac{1}{(k-1)}(1 - \mathcal{P}_{\text{PGM}})$$

$$B_i = \frac{1}{(k-1)}(\mathbb{1} - G_i) \; \forall \, i \in [k]$$

Inequalities

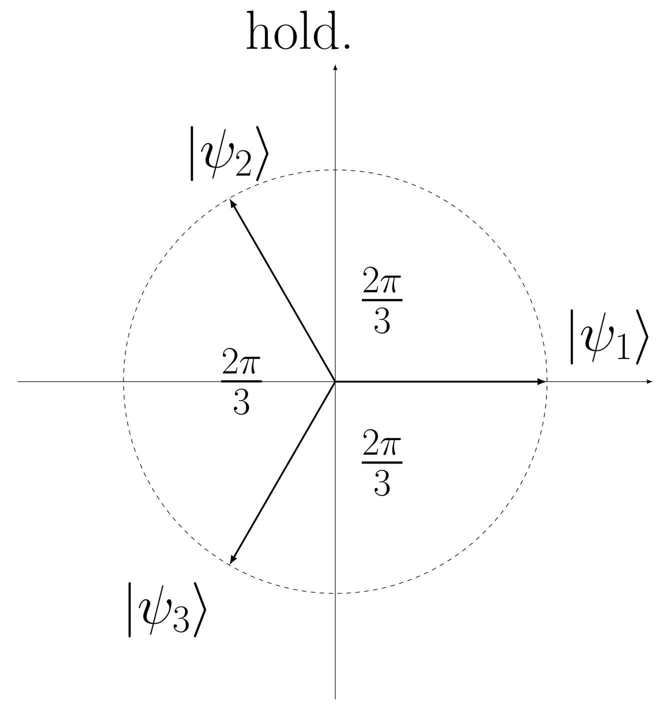
$$\mathcal{P}_{\text{best}} \geq \mathcal{P}_{\text{PGM}} \geq \frac{1}{k} + \frac{(1 - k\mathcal{P}_{\text{best}})^2}{k(k-1)}$$

$$\mathcal{P}_{\text{worst}} \leq \mathcal{P}_{\text{PBM}} \leq \frac{1}{k} - \frac{(1 - k\mathcal{P}_{\text{best}})^2}{k(k-1)^2}$$

Extreme cases

If $\mathcal{P}_{\text{PBM}} = \mathcal{P}_{\text{worst}}$, then $\mathcal{P}_{\text{PGM}} = \mathcal{P}_{\text{best}}$.

The converse does not necessarily hold.



- $\frac{2}{3} = \mathcal{P}_{\text{best}}, \mathcal{P}_{\text{PGM}}$
- $\frac{1}{3} = \mathcal{P}_{\text{blind}}$
- $\frac{1}{6} = \mathcal{P}_{\text{PBM}}$
- $0 = \mathcal{P}_{\text{worst}}$

Quantum State Exclusion

Helen selects ρ_i from $\{\rho_1, \rho_2, \ldots, \rho_k\}$ with probability p_i and sends it to Bob. Bob's task is to guess an index j that is NOT i.

$$\rho_i \longrightarrow$$

$$\longleftarrow j \neq i$$

If Bob can play this game perfectly, i.e, $\mathcal{P}_{\text{worst}} = 0$, then the set of states is said to be **antidistinguishable**.

Motivation

Set of states is perfectly distinguishable

$\Longleftrightarrow$

Set of states is pairwise orthogonal

Question

Is there an equivalent condition that can determine whether or not a set of states is **antidistinguishable**?

Reduced SDP

For a set of pure states

$$\mathcal{S} = \{|\psi_0\rangle, |\psi_1\rangle, \ldots, |\psi_{k-1}\rangle\},$$

we have the reduced primal-dual SDP pair:

Primal problem

$$\begin{aligned} \text{minimize} \quad & \sum_{i=0}^{k-1} \langle i | F_i | i \rangle \\ \text{subject to} \quad & \sum_{i=0}^{k-1} F_i = G \\ & F_i \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} \text{maximize} \quad & \text{Tr}(XG) \\ \text{subject to} \quad & X \leq |i\rangle\langle i| \end{aligned}$$

where G is their Gram matrix.

$(k-1)$-incoherence

X is $(k-1)$-incoherent if there exists a positive integer m and a set $\{v_0, \ldots, v_{m-1}\}$ with the property that each v_i has at most $(k-1)$ non-zero entries, and real scalars $c_0, c_1, \ldots, c_{m-1} \geq 0$ for which

$$X = \sum_{i=0}^{m-1} v_i v_i^*.$$

Connection between antidistinguishability and incoherence

Set of *pure* states is antidistinguishable

$\Longleftrightarrow$

Gram matrix is $(k-1)$- incoherent

Antidistinguishability bounds

Let $k \geq 2$ be an integer.

- Upper bounds: If
$$\sum_{i \neq j = 0}^{k-1} |\langle \psi_i | \psi_j \rangle| > k(k-2)$$
then $\mathcal{S}$ is not antidistinguishable. (See also [1].)
- Lower bounds: If
$$|\langle \psi_i | \psi_j \rangle| \leq \frac{1}{\sqrt{2}} \sqrt{\frac{k-2}{k-1}}$$
for all $0 \leq i \neq j \leq k-1$ then $\mathcal{S}$ is antidistinguishable.

Question

Is the sufficient condition tight for $n \geq 5$?

Reference

[1] Somshubhro Bandyopadhyay, Rahul Jain, Jonathan Oppenheim, and Christopher Perry. Conclusive exclusion of quantum states. *Physical Review A*, 89(2):022336, 2014.

- Quantum state exclusion through offset measurement. *Physical Review A, 110(4):042211, 2024.*
- Tight bounds for antidistinguishability and circulant sets of pure quantum states. *Quantum, 9:1622, 2025.*

What does this SDP tell us?

The dimension of the states is irrelevant for the pure states case, the number of states and their inner products suffice to determine antidistinguishability.