Numerical tools for extended nonlocal games

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Outline

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Extended nonlocal games

Bounding the values of extended nonlocal games

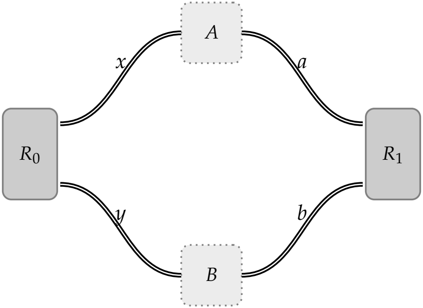
Monogamy-of-Entanglement games

Open questions

Nonlocal games

Nonlocal games

A *nonlocal game* is a cooperative game played between *Alice* and *Bob* against a *referee*.



1. Question and answer sets: (Σ_A, Σ_B) and (Γ_A, Γ_B),
2. Distributions on question pairs: $\pi : \Sigma_A \times \Sigma_B \to [0, 1]$,
3. A predicate $V : \Gamma_A \times \Gamma_B \times \Sigma_A \times \Sigma_B \to \{0, 1\}$, where

$$V(a, b|x, y) = \begin{cases} 1 & \text{if Alice and Bob win,} \\ 0 & \text{if Alice and Bob lose.} \end{cases}$$

Strategies and values for nonlocal games

Alice and Bob could use different types of *strategies*:

- *Classical strategies:* Alice and Bob answer deterministically, determined by functions of $f : \Sigma_\mathsf{A} \rightarrow \Gamma_\mathsf{A}$ and $g : \Sigma_\mathsf{B} \rightarrow \Gamma_\mathsf{B}$.

- *Quantum strategies:* Alice and Bob share a joint quantum system $\rho \in \mathrm{D}(\mathcal{A} \otimes \mathcal{B})$ and allow their answers to be outcomes of measurements on this shared system.

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The *value* of a nonlocal game is the maximal winning probability for the players to win over all strategies of a specified type.

For a nonlocal game, G, we denote the classical and quantum values as

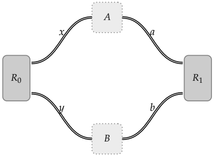
- Classical value: $\omega(G)$,
- Quantum value: $\omega^*(G)$.

Example: The CHSH game

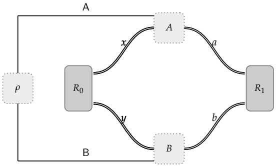
The CHSH game (G_{CHSH}). Question and answer sets over $\{0, 1\}$.
Question pairs $\{00, 01, 10, 11\}$ selected with equal probability.
Winning condition iff $a \oplus b = x \wedge y$.

$$\omega(G_{\text{CHSH}}) < \omega^*(G_{\text{CHSH}})$$

- $\omega(G_{\text{CHSH}}) = \frac{3}{4} = 0.75$:



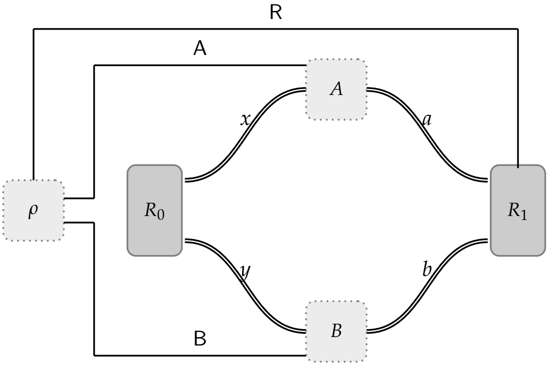
- $\omega^*(G_{\text{CHSH}}) = \cos^2(\frac{\pi}{8}) \approx 0.8536$:



Extended nonlocal games

Extended nonlocal games

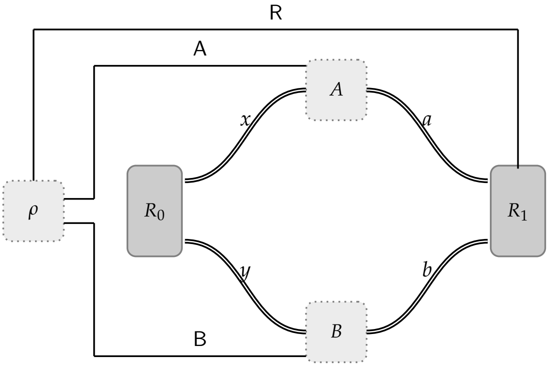
An *extended nonlocal game* (ENLG) is specified by:



- A probability distribution $\pi : X \times Y \rightarrow [0, 1]$ for alphabets X and Y.
- A collection of measurement operators $\{P_{a,b,x,y} : a \in A,\ b \in B,\ x \in X,\ y \in Y\} \subset \mathrm{Pos}(\mathcal{R})$ where $\mathcal{R}$ is the space corresponding to R and A, B are alphabets.

Extended nonlocal games

An (ENLG) is played in the following manner:



1. Alice and Bob present referee with register R.
2. Referee generates $(x, y) \in X \times Y$ according to π and sends x to Alice and y to Bob. Alice responds with a and Bob with b.
3. Referee measures R w.r.t. measurement $\{P_{a,b,x,y}, \mathbb{1} - P_{a,b,x,y}\}$. Outcome is either *loss* or *win*.

Strategies for extended nonlocal games

One may consider *strategies* for Alice and Bob in an ENLG:

- *Standard quantum strategies*:
  - $\sigma \in \mathrm{D}(\mathcal{U} \otimes \mathcal{R} \otimes \mathcal{V})$.
  - $\{A_a^x : a \in A\} \subset \mathrm{Pos}(\mathcal{U})$ and $\{B_b^y : b \in B\} \subset \mathrm{Pos}(\mathcal{V})$.
- *Unentangled strategies*: Standard quantum strategy where:
  - σ is separable.
- *Commuting measurement strategies*: Standard quantum strategy where:
  - $\sigma \in \mathrm{D}(\mathcal{R} \otimes \mathcal{H})$,
  - $[A_a^x, B_b^y] = 0$ for all x, y, a, b.
- *Non-signaling strategies*:
  - Satisfies non-signaling constraints.

Values of extended nonlocal games

The *value* of an ENLG, G, is the maximal winning probability for the players to win over all strategies of a specified type:

- *Unentangled*: $\omega(G)$,
- *Standard quantum*: $\omega^*(G)$,
- *Commuting measurement*: $\omega_{\mathsf{c}}(G)$,
- *Non-signaling*: $\omega_{\mathsf{ns}}(G)$.

The values obey the following relationship:

$$0 \leq \omega(G) \leq \omega^*(G) \leq \omega_{\mathsf{c}}(G) \leq \omega_{\mathsf{ns}}(G) \leq 1.$$

Bounding the values of extended nonlocal games

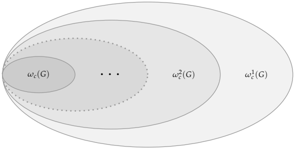
Calculating values of extended nonlocal games

One may either directly calculate or bound the value of extended nonlocal games:

- $\omega(G)$: A closed form expression exists that allows one to directly calculate this value.
- $\omega_{\mathsf{ns}}(G)$: May be phrased as an semidefinite program.

Calculating standard quantum values of extended nonlocal games

- The *extended NPA hierarchy*: extension of the NPA hierarchy[1,2] that may be used to upper bound the standard quantum value for ENLGs.



- $\omega^*(G)$: Extended NPA hierarchy to upper bound. May also adapt "see-saw" method[3] for lower bounds.

- $\omega_c(G)$: Extended NPA hierarchy.

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[1] Doherty, Liang, Toner, Wehner: "The quantum moment problem and bounds on entangled multi-prover games", (2008).

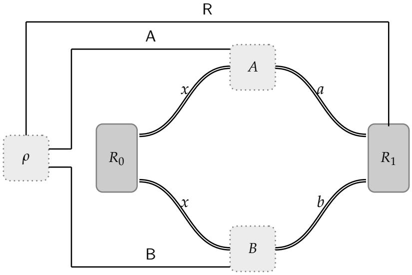
[2] Navascues, Pironio, Acin: "A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations", (2008).

[3] Liang, Doherty: "Bounds on Quantum Correlations in Bell Inequality Experiments", (2007).

Monogamy-of-Entanglement games

Monogamy-of-entanglement games

Monogamy-of-entanglement games[4], are a special type of extended nonlocal game.



1. Same question and answer sets: X and A.
2. Alice and Bob get the same question: $\pi(x, y) = 0$ for $x \neq y$.
3. Referee's measurement operator: $P : A \times X \to \mathrm{Pos}(\mathcal{R})$.
4. Winning condition: Iff Alice's output, Bob's output, and the referee's output are all the *equal*.

[4] Tomamichel, Fehr, Kaniewski, Wehner : "A Monogamy-of-Entanglement Game With Applications to Device-Independent Quantum Cryptography", (2013).

The BB84 monogamy-of-entanglement game

The BB84 game (G_{BB84} for short)[5] is defined by:

1. Question and answer sets:

$$\Sigma = \Gamma = \{0, 1\},$$

2. Uniform probability for questions:

$$\pi(0) = \pi(1) = \frac{1}{2}$$

3. Measurements defined by the BB84 bases:

$$\text{For } x = 0: \quad R(0|0) = |0\rangle\langle 0|, \qquad R(1|0) = |1\rangle\langle 1|$$
$$\text{For } x = 1: \quad R(0|1) = |+\rangle\langle +|, \quad R(1|1) = |-\rangle\langle -|$$

The *unentangled* and *standard quantum* values for G_{BB84} coincide:

$$\omega(G_{BB84}) = \omega^*(G_{BB84}) = \cos^2(\pi/8) \approx 0.8536$$

[5] G_{BB84} was introduced in [Tomamichel, Fehr, Kaniewski, Wehner, (2013)].

A natural question for monogamy-of-entanglement games

- *Question:* For any monogamy-of-entanglement game, G, is it true that the *unentangled* and *standard quantum* values always coincide? In other words is it true that:

  $$\omega(G) = \omega^*(G)$$

  for all monogamy-of-entanglement games G?

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  for all monogamy-of-entanglement games G?

- *Answer:*
  - For certain cases: Yes.
  - In general: No.

$$\omega(G) = \omega^*(G)$$

In general No

Monogamy-of-entanglement games where $\omega(G) \neq \omega^*(G)$

There exists a monogamy-of-entanglement game, G, with $|\Sigma| = 4$ and $|\Gamma| = 3$ such that

$$\omega(G) < \omega^*(G).$$

1. Question and answer sets:

$$\Sigma = \{0, 1, 2, 3\}, \quad \Gamma = \{0, 1, 2\}.$$

2. Uniform probability for questions:

$$\pi(0) = \pi(1) = \pi(2) = \pi(3) = \frac{1}{4}.$$

3. Measurements defined by a mutually unbiased basis[6]:

$$\{R(0|x), R(1|x), R(2|x)\}.$$

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[6] $|u_x(a)^* u_{x'}(a)|^2 = 1/|\Gamma|$ for $R(a|x) = u_x(a)u_x(a)^*, R(a|x') = u_{x'}(a)u_{x'}(a)^*$

Monogamy-of-entanglement games where $\omega(G) \neq \omega^*(G)$

- An exhaustive search over all unentangled strategies reveals an optimal unentangled value:

$$\omega(G) = \frac{3 + \sqrt{5}}{8} \approx 0.6545.$$

- Alternatively, a computer search over standard quantum strategies and a heuristic approximation for the upper bound of $\omega^*(G)$ reveals that:

$$2/3 \geq \omega^*(G) \geq 0.6609.$$

$$\omega(G) = \omega^*(G)$$

For certain classes, Yes.

Monogamy games that obey $\omega(G) = \omega^*(G)$

Theorem (Johnston, Mittal, R, Watrous)

For any monogamy-of-entanglement game, G, for which $|\Sigma| = 2$:

$$\omega(G) = \omega^*(G).$$

Open questions

Unentangled vs. standard quantum strategies for monogamy-of-entanglement games

Inputs ($\|\Sigma\|$)	Outputs ($\|\Gamma\|$)	$\omega^*(G) = \omega(G)$	$\omega^*(G^n) = \omega^*(G)^n$	$\omega_{ns}(G^n) = \omega_{ns}(G)^n$
2	$\|\Gamma\| \geq 1$	yes	yes[7]	no
3	$\|\Gamma\| \geq 1$	?	?	no
4	3	no	?	no

Question: What about $|\Sigma| = 3$?

- Proof technique fails for $|\Sigma| > 2$.
- Computational search:
  - Generate random monogamy-of-entanglement games where $|\Sigma| = 3$ and $|\Gamma| \geq 2$.
  - 10^8 random games generates, no counterexamples found.

[7] So long as the measurements used by the referee are projective and the probability distribution, π, from which the questions are asked is uniform.

Other questions

- Closed-form equation for monogamy-of-entanglement games when the questions are selected with non-uniform probability?
- Further development of numerical tools to study extended nonlocal games.
- Extended nonlocal games as a tool to study steering, device independnet cryptography, etc.

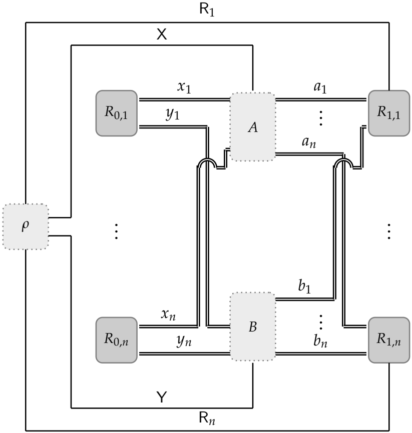
Thanks!

Thank you for your attention!

Parallel repetition of monogamy-of-entanglement games

Parallel repetition of monogamy-of-entanglement games

- *Parallel repetition:* Run a monogamy-of-entanglement game, G, for n times in parallel, denoted as G^n.
- *Strong parallel repetition:* $\omega(G^n) = \omega(G)^n$



Question: Do all monogamy-of-entanglement games obey strong parallel repetition?

Parallel repetition of monogamy-of-entanglement games

- Recall:

$$\omega(G_{\text{BB84}}) = \omega^*(G_{\text{BB84}}) = \cos^2(\pi/8) \approx 0.8536.$$

- G_{BB84} obeys strong parallel repetition[8]:

$$\omega^*(G^n_{\text{BB84}}) = \omega^*(G_{\text{BB84}})^n = \left(\cos^2(\pi/8)\right)^n.$$

[8] [Tomamichel, Fehr, Kaniewski, Wehner, (2013)]

Further properties of monogamy-of-entanglement games

General properties about monogamy-of-entanglement games:[9]

- For any monogamy-of-entanglement game, G, for which $|X| = 2$:

$$\omega(G) = \omega^*(G).$$

[9] Johnston, Mittal, R., Watrous: "Extended nonlocal games and monogamy-of-entanglement games", (2015).

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Parallel repetition of monogamy-of-entanglement games

Parallel repetition of monogamy-of-entanglement games:[10]

- Let $G = (\pi, P)$ be a monogamy game where $|X| = 2$, π is uniform over X, and $P_{a,x}$ are projective operators. It holds that for all n:

$$\omega^*(G^n) = \left(\frac{1}{2} + \frac{1}{2}\sqrt{c(G)}\right)^n,$$

where $c(G)$ is the maximal overlap of the referee's measurements:

$$c(G) = \max_{\substack{x,y \in X \\ x \neq y}} \max_{a,b \in A} \left\| \sqrt{P_{a,x}} \sqrt{P_{b,y}} \right\|^2.$$

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- There exists a monogamy-of-entanglement game, G, such that

$$\omega_{\text{ns}}(G^2) \neq \omega_{\text{ns}}(G)^2.$$

[10] Johnston, Mittal, R., Watrous: "Extended nonlocal games and monogamy-of-entanglement games", (2015).