Quantum nonlocality without entanglement and state discrimination measures

Shayeef Murshid * Tathagata Gupta † Vincent Russo ‡ and
Somshubhro Bandyopadhyay §

June 26, 2025

Abstract

An ensemble of product states is said to exhibit "quantum nonlocality without entanglement" if the states cannot be optimally discriminated by local operations and classical communication (LOCC). We show that this property can depend on the measure of state discrimination. We present a family of ensembles, each consisting of six linearly independent, equally probable product states for which LOCC fails to achieve optimal minimum-error discrimination but succeeds in achieving optimal unambiguous discrimination.

1 Introduction

Composite quantum systems may exhibit nonlocal properties. For example, the celebrated Bell nonlocality [1, 2] arises from entangled states through violations of Bell-type inequalities. Somewhat less known, though well-studied, is quantum nonlocality without entanglement [3]. This nonlocality, which may be viewed as dual to the Bell type, manifests in state discrimination problems in the distant-lab paradigm of quantum information theory.

Suppose two distant observers, Alice and Bob, share a state chosen from an ensemble of product states

$$\mathcal{E}_\psi = \{(\eta_i, |\psi_i\rangle = |a_i\rangle \otimes |b_i\rangle) : i = 1, \ldots, N\}, \tag{1}$$

where η_i is the prior probability associated with $|\psi_i\rangle$. They have complete knowledge of $\mathcal{E}_\psi$ but do not know which particular $|\psi_i\rangle$ they share. Their objective is to determine this "unknown" state as well as possible.

*Electronics and Communication Sciences Unit, Indian Statistical Institute, 203 B. T. Road, Kolkata 700108, India. shayeef.murshid91@gmail.com

†Physics and Applied Mathematics Unit, Indian Statistical Institute, 203 B. T. Road, Kolkata 700108, India. tathagatagupta@gmail.com

‡Unitary Foundation, 315 Montogomery St, Fl 10 San Francisco, California 94104, USA. vincent@unitary.foundation

§Department of Physical Sciences, Bose Institute, EN 80, Bidhannagar, Kolkata 700091, India, som@jcbose.ac.in

The task of determining an unknown state chosen from a known set of states is the standard state discrimination problem. However, in this case, Alice and Bob being physically separated cannot perform any measurement of their choice. The only measurements they can implement belong to the class of *local operations and classical communication* (LOCC); that is, they are free to perform quantum operations on their local systems and communicate via classical channels but cannot exchange quantum systems. Thus, the question is: Can they optimally discriminate the given states by LOCC? By optimal discrimination, we mean attaining the global optimum corresponding to a measure of state discrimination; in particular, if the given states are orthogonal, optimal implies perfect discrimination [4, 5, 6].

One might expect optimal discrimination of product states, parts of which may have been prepared separately, would be possible by LOCC. However, it turns out this is not always the case [7]. There exist product ensembles, members of which cannot be optimally discriminated by LOCC, even when they are mutually orthogonal [3]. Such ensembles are said to exhibit *nonlocality without entanglement*. Here, nonlocality refers to the fact that joint measurements can extract more information about the quantum state than LOCC protocols.

One could express this nonlocality in terms of a well-defined state discrimination measure, such as probability of success or fidelity [8]. Let f be such a measure. For an ensemble $\mathcal{E}_\psi$ and a measurement $\mathbb{M}$ (a POVM), the quantity $0 \leqslant f\left(\mathcal{E}_\psi;\mathbb{M}\right) \leqslant 1$ then tells us how well the members of $\mathcal{E}_\psi$ can be discriminated by $\mathbb{M}$.

Let $f_{G}\left(\mathcal{E}_\psi\right)$ and $f_{L}\left(\mathcal{E}_\psi\right)$ be the respective global and local optimum, defined as

$$f_{G}\left(\mathcal{E}_\psi\right) = \sup_{\mathbb{M}} f\left(\mathcal{E}_\psi;\mathbb{M}\right) \quad \text{and} \quad f_{L}\left(\mathcal{E}_\psi\right) = \sup_{\mathbb{M}\in\text{LOCC}} f\left(\mathcal{E}_\psi;\mathbb{M}\right). \tag{2}$$

Since LOCC is a strict subset of all measurements one may perform on a composite system, we have

$$f_{L}\left(\mathcal{E}_\psi\right) \leqslant f_{G}\left(\mathcal{E}_\psi\right). \tag{3}$$

Therefore, if $\mathcal{E}_\psi$ is nonlocal in the sense discussed above, the above inequality must be strict

$$f_{L}\left(\mathcal{E}_\psi\right) < f_{G}\left(\mathcal{E}_\psi\right). \tag{4}$$

To check whether a given product ensemble is nonlocal or not, one therefore needs to compute both optima explicitly. However, for orthogonal states, computing the local optimum may not be necessary as long as one can show whether or not the states can be perfectly discriminated by LOCC.

Suppose the states $|\psi_i\rangle \in \mathcal{E}_\psi$ form an orthonormal basis. Then perfect discrimination is always possible by a separable measurement [9], the measurement operators being $|\psi_i\rangle\langle\psi_i| = |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|$, $i = 1,\ldots,N$. Nevertheless, there exist orthogonal product bases, elements of which cannot be perfectly discriminated by LOCC [3]. This shows that LOCC is a strict subset of separable measurements (SEP), which, of course, is a strict subset of global measurements [10]. This can be conveniently expressed as

$$\text{LOCC} \subset \text{SEP} \subset \text{GLOBAL}. \tag{5}$$

Thus for any product ensemble $\mathcal{E}_\psi$ (or for any ensemble, for that matter) it holds that

$$f_{L}\left(\mathcal{E}_\psi\right) \leqslant f_{S}\left(\mathcal{E}_\psi\right) \leqslant f_{G}\left(\mathcal{E}_\psi\right), \tag{6}$$

where $f_{S}\left(\mathcal{E}_{\psi}\right)=\sup_{\mathbb{M}\in\mathrm{SEP}} f\left(\mathcal{E}_{\psi};\mathbb{M}\right)$. If at least one of the above inequalities is strict, then $\mathcal{E}_{\psi}$ is nonlocal. For example, if $\mathcal{E}_{\psi}$ is an orthonormal product basis and is nonlocal, it implies that

$$f_{L}\left(\mathcal{E}_{\psi}\right)<f_{S}\left(\mathcal{E}_{\psi}\right)=f_{G}\left(\mathcal{E}_{\psi}\right)=1. \tag{7}$$

Motivation

In this paper we will consider measures corresponding to minimum-error [11] and unambiguous discrimination [12, 13, 14]. The former minimizes the average error and applies to any set of states; the corresponding measure is the success probability, the maximum probability that the unknown state is correctly determined [15]. The latter strategy, however, can only be applied to sets of states satisfying a certain condition. If this condition is met, the unknown state can be determined, without error, with a nonzero probability; for example, a set of pure states can be unambiguously discriminated if and only if they are linearly independent [16]. Note that in the minimum-error case, the conclusion could be erroneous, whereas in the latter, there's no room for error – a measurement outcome either correctly identifies the state or is inconclusive.

Once an ensemble is specified, one might expect the relationships between different measurement optimas to be measure-agnostic. However, this turns out not to be the case. Consider the double trine ensemble consisting of three equiprobable two-qubit product states [7]:

$$\mathcal{E}_{\alpha}=\left\{\left(\frac{1}{3},\left|\alpha_{i}\right\rangle\otimes\left|\alpha_{i}\right\rangle\right):i=1,2,3\right\}, \tag{8}$$

where

$$\left|\alpha_{1}\right\rangle=\left|0\right\rangle,\quad\left|\alpha_{2}\right\rangle=-\frac{1}{2}\left|0\right\rangle-\frac{\sqrt{3}}{2}\left|1\right\rangle,\quad\left|\alpha_{3}\right\rangle=-\frac{1}{2}\left|0\right\rangle+\frac{\sqrt{3}}{2}\left|1\right\rangle. \tag{9}$$

Let $p^{me}\left(\mathcal{E}_{\alpha}\right)$ and $p^{ud}\left(\mathcal{E}_{\alpha}\right)$ denote the (optimum) success probabilities for minimum-error and unambiguous discrimination, respectively. Then it holds that [17, 18]

$$p_{L}^{me}\left(\mathcal{E}_{\alpha}\right)<p_{S}^{me}\left(\mathcal{E}_{\alpha}\right)=p_{G}^{me}\left(\mathcal{E}_{\alpha}\right), \tag{10}$$

$$p_{L}^{ud}\left(\mathcal{E}_{\alpha}\right)=p_{S}^{ud}\left(\mathcal{E}_{\alpha}\right)<p_{G}^{ud}\left(\mathcal{E}_{\alpha}\right). \tag{11}$$

Note that $\mathcal{E}_{\alpha}$ exhibits nonlocality without entanglement for both measures. However, the relationships between measurement optimas are different. In the minimum-error case, LOCC is suboptimal to SEP which achieves the global optimum; in the unambiguous case, LOCC and SEP are equally good but remain suboptimal to global measurements. This led the authors of [18] to ask the following question:

Do product ensembles exist that exhibit nonlocality without entanglement for one discrimination measure but not for another?

In this paper, we provide a positive answer to this question. Specifically, we construct a family of product ensembles, each ensemble consisting of six linearly independent equally probable product states for which LOCC fails to achieve optimal minimum-error discrimination but achieves optimal unambiguous discrimination. Thus, these ensembles exhibit nonlocality without entanglement for minimum-error discrimination but not for unambiguous discrimination.

2 Results

Consider the following set of states

$$\mathcal{S} = \left\{ |\psi_i\rangle \in \mathbb{C}^3 : i = 1,2,3,\ \langle \psi_i | \psi_j \rangle = s \in (0,1) \text{ for } i \neq j \right\}. \tag{12}$$

The above set of states is linearly independent. This follows from the result that a set of $N \leqslant d$ pure states in $\mathbb{C}^d$ with pairwise real and equal inner products is linearly independent if and only if the inner product lies in the interval $\left(-\frac{1}{N-1}, 1\right)$ [19].

Define the following set of product states

$$\mathcal{T} = \left\{ |\psi_i\rangle \otimes |\psi_j\rangle \in \mathbb{C}^3 \otimes \mathbb{C}^3 : |\psi_i\rangle, |\psi_j\rangle \in \mathcal{S},\ i \neq j,\ i,j = 1,2,3 \right\}. \tag{13}$$

The set $\mathcal{T}$ is also linearly independent. That is because its elements can be unambiguously discriminated, and hence, must be linearly independent (a given set of pure states can be unambiguously discriminated if and only if they are linearly independent) [16].

We study minimum-error and unambiguous discrimination of the members of the ensemble

$$\mathcal{E}_{\mathcal{T}} = \left\{ \left(\frac{1}{6}, |\psi_i\rangle \otimes |\psi_j\rangle \right) : i \neq j,\ i,j = 1,2,3 \right\}. \tag{14}$$

The main result is stated as follows.

Theorem 1. *Let $p^{me}(\mathcal{E}_{\mathcal{T}})$ and $p^{ud}(\mathcal{E}_{\mathcal{T}})$ denote the respective (optimum) success probabilities for minimum-error and unambiguous discrimination of the members of $\mathcal{E}_{\mathcal{T}}$. Then*

$$p_{L}^{me}(\mathcal{E}_{\mathcal{T}}) < p_{G}^{me}(\mathcal{E}_{\mathcal{T}}), \tag{15}$$

$$p_{L}^{ud}(\mathcal{E}_{\mathcal{T}}) = p_{G}^{ud}(\mathcal{E}_{\mathcal{T}}). \tag{16}$$

Thus $\mathcal{E}_{\mathcal{T}}$ is nonlocal with respect to minimum-error discrimination but not unambiguous discrimination.

To prove (15), we will first convert the problem of optimal LOCC discrimination of the members of $\mathcal{E}_{\mathcal{T}}$ to that of perfect LOCC discrimination of a set of orthogonal states using a theorem in [17]. We will then show that LOCC cannot perfectly discriminate these orthogonal states and hence, the members of $\mathcal{E}_{\mathcal{T}}$ cannot be optimally discriminated by LOCC. Thus the local optimum is strictly less than the global optimum in the minimum-error case. To prove (16), we will compute the global optimum explicitly and show that this is achievable by LOCC.

3 Minimum-error discrimination of $\mathcal{E}_{\mathcal{T}}$

Given an ensemble $\mathcal{E}_{\rho} = \{\eta_i, \rho_i\}_{i=1}^{N}$ and a measurement $\mathbb{M} = \{M_1, \ldots, M_N\}$, which is a collection of positive operators forming a resolution of the identity, the error probability is given by

$$p_{error}(\mathcal{E}_{\rho}, \mathbb{M}) = \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \eta_i \operatorname{Tr}(M_j \rho_i). \tag{17}$$

In minimum-error discrimination, the goal is to find a measurement that minimizes $p_{error}\left(\mathcal{E}_\rho, \mathbb{M}\right)$, thereby maximizing the success probability. The minimum-error probability is given by

$$p_{error}\left(\mathcal{E}_\rho\right) = \min_{\mathbb{M}} p_{error}\left(\mathcal{E}_\rho, \mathbb{M}\right) \tag{18}$$

and the success probability

$$p^{me}\left(\mathcal{E}_\rho\right) = 1 - p_{error}\left(\mathcal{E}_\rho\right). \tag{19}$$

The advantage of this approach is that it applies to any set of states. Finding the optimal solution for an arbitrary ensemble, however, is hard. Nevertheless, if the given states are pure and linearly independent, as in our case, an optimal measurement consists of orthonormal, rank one projectors [20]. This result was subsequently strengthened by the following theorem.

Theorem 2 ([17])**.** *Let $\mathcal{E}_\chi = \{\eta_i, |\chi_i\rangle\}_{i=1}^N$ be an ensemble of linearly independent pure states spanning a space $\mathcal{Y}$. There exists a unique orthonormal basis $\{|\xi_i\rangle\}_{i=1}^N$ of $\mathcal{Y}$ such that a measurement that achieves optimal minimum-error discrimination of the members of $\mathcal{E}_\chi$ also perfectly discriminates the states $|\xi_1\rangle, \ldots, |\xi_N\rangle$ and vice versa.*

The above theorem reduces the problem of optimal minimum-error discrimination of linearly independent pure states to that of perfect discrimination of mutually orthonormal pure states. We will make use of this fact to prove (15).

3.1 Proof of LOCC suboptimality

The following proposition follows from Theorem 2.

Proposition 3. *Let $\mathcal{W}$ be the subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3$ spanned by the elements of $\mathcal{T}$. There exists a unique orthonormal basis $\tilde{\mathcal{B}}$ of $\mathcal{W}$ such that a measurement that achieves the global optimum $p^{me}_{G}\left(\mathcal{E}_\mathcal{T}\right)$ perfectly discriminates the elements of $\tilde{\mathcal{B}}$ and vice versa.*

Corollary 4. *LOCC achieves the global optimum $p^{me}_{G}\left(\mathcal{E}_\mathcal{T}\right)$ if and only if the elements of $\tilde{\mathcal{B}}$ can be perfectly discriminated by LOCC.*

In what follows, we will find this unique orthonormal basis and show that its elements cannot be perfectly discriminated by LOCC. Therefore from Corollary 4, optimal minimum-error discrimination of the elements of $\mathcal{E}_\mathcal{T}$ is not possible by LOCC.

For ease of understanding, we will use the following notation to represent the elements of $\mathcal{T}$:

$$\begin{aligned}
|\phi_1\rangle &= |\psi_1\rangle \otimes |\psi_2\rangle & \quad |\phi_2\rangle &= |\psi_1\rangle \otimes |\psi_3\rangle \\
|\phi_3\rangle &= |\psi_2\rangle \otimes |\psi_1\rangle & \quad |\phi_4\rangle &= |\psi_2\rangle \otimes |\psi_3\rangle \\
|\phi_5\rangle &= |\psi_3\rangle \otimes |\psi_1\rangle & \quad |\phi_6\rangle &= |\psi_3\rangle \otimes |\psi_2\rangle
\end{aligned}$$

and therefore our ensemble can be written as $\mathcal{E}_\mathcal{T} = \left\{\frac{1}{6}, |\phi_i\rangle\right\}_{i=1}^{6}$.

Since the states $|\phi_i\rangle$ are linearly independent, the optimal measurement on $\mathcal{W}$ consists of orthogonal rank-one projectors [20]. Let this measurement be $\{E_i = |e_i\rangle\langle e_i|\}_{i=1}^{6}$, where $\mathrm{Tr}\left(E_i E_j\right) =$

δ_{ij} for $i, j = 1, \ldots, 6$, and $\sum_{i=1}^{6} E_i = \mathbf{1}_{\mathcal{W}}$. Thus the optimal measurement on $\mathbb{C}^3 \otimes \mathbb{C}^3$ is given by $\{E_1, \ldots, E_6, (\mathbf{1}_{3\times 3} - \mathbf{1}_{\mathcal{W}})\}$, where $(\mathbf{1}_{3\times 3} - \mathbf{1}_{\mathcal{W}})$ is the projector onto $\mathcal{W}^{\perp}$. It follows that $\tilde{\mathcal{B}} = \{|e_1\rangle, \ldots, |e_6\rangle\}$ must be the unique orthonormal basis of $\mathcal{W}$ mentioned in Prop. 3.

Our objective is to find the measurement $\{E_i = |e_i\rangle\langle e_i|\}_{i=1}^{6}$ as it would immediately lead to $\tilde{\mathcal{B}}$. Fortunately, this measurement turns out to be the well-known square-root measurement (SRM), also known as pretty-good measurement [21]. The SRM operators are one-dimensional projectors [22, 23, 24]

$$\mu_i = |\mu_i\rangle\langle\mu_i|, \quad i = 1, \ldots, 6 \tag{20}$$

satisfying $\langle \mu_i | \mu_j \rangle = \delta_{i,j}$ for all $i, j = 1, \ldots, 6$ and $\sum_{i=1}^{6} \mu_i = \mathbf{1}_{\mathcal{W}}$, where

$$|\mu_i\rangle = \rho^{-1/2} \frac{1}{\sqrt{6}} |\phi_i\rangle \quad \text{ and } \quad \rho = \frac{1}{6} \sum_{i=1}^{6} |\phi_i\rangle\langle\phi_i|. \tag{21}$$

Note that the vectors $\{|\mu_i\rangle\}_{i=1}^{6}$ form an orthonormal basis of $\mathcal{W}$.

Theorem 5. *The square-root measurement is optimal for minimum-error discrimination of the members of* $\mathcal{E}_{\mathcal{T}} = \left\{ \frac{1}{6}, |\phi_i\rangle \right\}_{i=1}^{6}$.

Proof. We know that for minimum-error discrimination of a set of linearly independent pure states, the SRM is optimal when all the diagonal elements of the square root of the Gram matrix of the states are equal [22]. A straightforward calculation shows this is indeed the case (proof in Appendix C). Therefore our claim is proved. □

We therefore have $\tilde{\mathcal{B}} = \{|\mu_1\rangle, \ldots, |\mu_6\rangle\}$. Note that while $|\phi_i\rangle$ are product vectors, $|\mu_i\rangle$ may not be. We now prove that the vectors $|\mu_1\rangle, \ldots, |\mu_6\rangle$ cannot be perfectly discriminated by LOCC.

First we prove the following property of $\mathcal{W}$.

Lemma 6. *$\mathcal{W}$ does not admit an orthogonal product basis.*

Proof. The proof is by contradiction. Assume that $\mathcal{B}_{\text{OPB}}$ is an orthogonal product basis of $\mathcal{W}$. Let $|\psi_1'\rangle$ be a unit vector orthogonal to $|\psi_2\rangle$ and $|\psi_3\rangle$. Since the subspace orthogonal to the span of $\{|\psi_2\rangle, |\psi_3\rangle\}$ is one-dimensional, this vector $|\psi_1'\rangle$ is unique.

Let us now assume that $|\alpha\rangle \otimes |\beta\rangle$ is a product state in the subspace orthogonal to that spanned by $\mathcal{B}_{\text{OPB}} \cup \{|\psi_1'\rangle \otimes |\psi_1'\rangle\}$. Therefore, $|\alpha\rangle \otimes |\beta\rangle$ is orthogonal to every member of the set

$$\mathcal{A} = \{|\psi_1'\rangle \otimes |\psi_1'\rangle, |\psi_i\rangle \otimes |\psi_j\rangle : i, j = 1, 2, 3,\ i \neq j\}. \tag{22}$$

This can happen if $|\alpha\rangle$ is orthogonal to Alice's state or $|\beta\rangle$ is orthogonal to Bob's state, or both. Since $\mathcal{A}$ contains seven states, this means there are at least four states where either $|\alpha\rangle$ is orthogonal to Alice's state (and $|\beta\rangle$ is orthogonal to the rest) or $|\beta\rangle$ is orthogonal to Bob's state (and $|\alpha\rangle$ is orthogonal to the rest).

We first consider the case where $|\alpha\rangle$ is orthogonal to Alice's side in exactly four states of $\mathcal{A}$ and show that this is impossible.

1. **Subcase 1.1:** $|\alpha\rangle$ is nonorthogonal to $|\psi_1'\rangle$.

   Given that $|\alpha\rangle$ cannot be orthogonal to all states in $\{|\psi_i\rangle\}_{i=1}^3$, it will be orthogonal to exactly two of them. Let them be $|\psi_1\rangle$ and $|\psi_2\rangle$. This means $|\beta\rangle$ is orthogonal to $\{|\psi_1'\rangle, |\psi_1\rangle, |\psi_2\rangle\}$. We now show that the set $\{|\psi_1'\rangle, |\psi_1\rangle, |\psi_2\rangle\}$ is linearly independent and as a result $|\beta\rangle$ cannot be orthogonal to all of them.

   Let $|\psi_1'\rangle = a|\psi_1\rangle + b|\psi_2\rangle + c|\psi_3\rangle$ and note that when $c \neq 0$, the set $\{|\psi_1'\rangle, |\psi_1\rangle, |\psi_2\rangle\}$ is linearly independent. Therefore when $c = 0$,

   $$\begin{aligned} \langle\psi_2|\psi_1'\rangle &= as + b = 0, \text{ and} \\ \langle\psi_3|\psi_1'\rangle &= as + bs = 0. \end{aligned} \tag{23}$$

   Since $s \in (0.1)$, it follows that $b = 0$. This implies $|\psi_1'\rangle$ and $|\psi_1\rangle$ are linearly dependent, which is a contradiction. Therefore, c is necessarily nonzero and as a result $\{|\psi_1'\rangle, |\psi_1\rangle, |\psi_2\rangle\}$ is linearly independent. Note that a similar argument shows that $\{|\psi_1'\rangle, |\psi_1\rangle, |\psi_3\rangle\}$ is linearly independent.

2. **Subcase 1.2:** $|\alpha\rangle$ is orthogonal to $|\psi_1'\rangle$.

   In this case, $|\psi_1'\rangle \otimes |\psi_1'\rangle$ can be regarded as an element of the four-state set whose states on Alice's side are orthogonal to $|\alpha\rangle$. Therefore $|\alpha\rangle$ has to be orthogonal to at least two states in $\{|\psi_i\rangle\}_{i=1}^3$. From the previous case it can be noted that each of the sets $\{|\psi_1'\rangle, |\psi_1\rangle, |\psi_2\rangle\}$ and $\{|\psi_1'\rangle, |\psi_1\rangle, |\psi_3\rangle\}$ is linearly independent. The remaining set $\{|\psi_1'\rangle, |\psi_2\rangle, |\psi_3\rangle\}$ is linearly independent since $|\psi_1'\rangle$ is orthogonal to $|\psi_2\rangle$ and $|\psi_3\rangle$.

   Therefore, in both cases, $|\alpha\rangle$ needs to be orthogonal to a linearly independent spanning set of $\mathbb{C}^3$, which is impossible.

The remaining cases where $|\alpha\rangle$ is orthogonal to Alice's side in five or more states are easy to discard, since these require $|\alpha\rangle$ to be orthogonal to Alice's side in at least four states of $\mathcal{A}$. By symmetry, we can argue that there cannot be four or more states in $\mathcal{A}$ where $|\beta\rangle$ is orthogonal to Bob's side. Therefore, we have shown that there exists no product state orthogonal to the subspace spanned by the states $\mathcal{B}_{\text{OPB}} \cup \{|\psi_1'\rangle \otimes |\psi_1'\rangle\}$. Since $|\psi_1'\rangle \otimes |\psi_1'\rangle$ is orthogonal to the subspace $\mathcal{W}$, this means that $\mathcal{B}_{\text{OPB}} \cup \{|\psi_1'\rangle \otimes |\psi_1'\rangle\}$ forms an unextendible product basis (UPB). But we know that in $\mathbb{C}^3 \otimes \mathbb{C}^3$ any UPB contains exactly five elements [25, 26] and therefore we reach a contradiction. Consequently, there is no orthogonal product basis of $\mathcal{W}$. This completes our proof. □

Lemma 6 tells us that $\tilde{\mathcal{B}}$ cannot be an orthonormal product basis, which means at least one of its elements $|\mu_i\rangle$ must be entangled. We will now show that all elements of $\tilde{\mathcal{B}}$ are entangled and also have the same Schmidt rank.

Lemma 7. *$\tilde{\mathcal{B}} = \{|\mu_1\rangle, \ldots, |\mu_6\rangle\}$ is an entangled orthonormal basis of $\mathcal{W}$. Moreover, the vectors $|\mu_i\rangle$ have the same Schmidt rank.*

To prove this, we will show that the vectors $|\mu_i\rangle$ are connected by local unitaries. We suppress the $1/\sqrt{6}$ factor for convenience.

Proof. Let us write $|\mu_i\rangle$ as

$$|\mu_i\rangle = \rho^{-1/2}\,|\psi_{i_1}\rangle\,|\psi_{i_2}\rangle\,,\ \text{where } i_1 \neq i_2$$

and let $U_{i_1 j_1} : \mathbb{C}^3 \to \mathbb{C}^3$ be the unitary operator that satisfies

$$U_{i_1 j_1}\,|\psi_{i_1}\rangle = |\psi_{j_1}\rangle\,, U_{i_1 j_1}\,|\psi_{j_1}\rangle = |\psi_{i_1}\rangle$$

and is identity on the rest. Note that, since

$$\rho = \frac{1}{6}\sum_{i=1}^{6} |\phi_i\rangle\langle\phi_i|$$

we have

$$\left(U_{i_1 j_1} \otimes U_{i_2 j_2}\right)^{-1} \rho\,\left(U_{i_1 j_1} \otimes U_{i_2 j_2}\right) = \rho.$$

Thus $U_{i_1 j_1} \otimes U_{i_2 j_2}$ commutes with ρ and hence $\rho^{-1/2}$. Therefore,

$$\begin{aligned}
\left(U_{i_1 j_1} \otimes U_{i_2 j_2}\right)|\mu_i\rangle &= \left(U_{i_1 j_1} \otimes U_{i_2 j_2}\right)\rho^{-1/2}\,|\psi_{i_1}\rangle\,|\psi_{i_2}\rangle \\
&= \rho^{-1/2}\left(U_{i_1 j_1} \otimes U_{i_2 j_2}\right)|\psi_{i_1}\rangle\,|\psi_{i_2}\rangle \\
&= \rho^{-1/2}\,|\psi_{j_1}\rangle\,|\psi_{j_2}\rangle \\
&= |\mu_j\rangle\,.
\end{aligned}$$

This completes the proof. □

We now come to the main result of this section.

Lemma 8. *The elements of the basis $\tilde{\mathcal{B}} = \{|\mu_1\rangle, \ldots, |\mu_6\rangle\}$ cannot be perfectly discriminated by LOCC.*

The proof is given in Appendix B, a sketch of which is presented below.

We use a result by Chen *et. al.* [27] which provides a necessary condition for a set of orthogonal pure states to be LOCC distinguishable.

Lemma 9. *([27]) If the states $|\mu_1\rangle, \ldots, |\mu_6\rangle$ can be perfectly discriminated by LOCC, then for every $i \in \{1, \ldots, 6\}$ it holds that*

$$|\mu_i\rangle = \sum_j |\alpha_{ij}\rangle \otimes |\beta_{ij}\rangle\,, \tag{24}$$

where

$$\left(\langle\alpha_{im}| \otimes \langle\beta_{im}|\right)|\mu_i\rangle \neq 0 \text{ for all } i, \text{ and}, \tag{25}$$

$$\left(\langle\alpha_{jm}| \otimes \langle\beta_{jm}|\right)|\mu_i\rangle = 0 \text{ for all } m \text{ and } j \neq i. \tag{26}$$

We know that $|\mu_1\rangle, \ldots, |\mu_6\rangle$ are entangled. Now if they are LOCC distinguishable, there must be at least two linearly independent product vectors appearing in the decomposition of each $|\mu_i\rangle$ such that these two conditions are satisfied. In Appendix B we show that this is impossible.

As noted earlier, by virtue of Theorem 2, a measurement that achieves optimal minimum-error discrimination of the members of $\mathcal{E}_{\mathcal{T}}$ must perfectly discriminate the elements of the basis $\tilde{\mathcal{B}}$. Since LOCC cannot perfectly discriminate the elements of the basis $\tilde{\mathcal{B}}$, by Corollary 4 we have the following theorem:

Theorem 10. *The optimal minimum-error discrimination of the elements of $\mathcal{E}_{\mathcal{T}}$ cannot be achieved by LOCC.*

Therefore, $p_L^{me}(\mathcal{E}_{\mathcal{T}}) < p_G^{me}(\mathcal{E}_{\mathcal{T}})$, proving (15) of Theorem 1.

4 Unambiguous discrimination of $\mathcal{E}_{\mathcal{T}}$

Unambiguous discrimination (UD) [12, 13, 14] of the members of an ensemble $\mathcal{E}_\chi = \{\eta_i, |\chi_i\rangle\}_{i=1}^N$, involves constructing a measurement $E = \{E_i\}_{i=0}^N$ with $N+1$ outcomes such that

$$\mathrm{Tr}\left(E_i \left|\chi_j\middle\rangle\middle\langle\chi_j\right|\right) = p_i \delta_{ij} \tag{27}$$

for all $i, j \in \{1, \ldots, N\}$ and E_0 corresponds to the inconclusive outcome. Here p_i, called the *efficiency* for $|\chi_i\rangle$, denotes the probability that upon receiving the input state $|\chi_i\rangle$ the measurement successfully identifies it. When the efficiencies are demanded to be equal, i.e., $p_i = p$ for all $i = 1, \ldots, N$, the corresponding task is referred to as *equiprobable unambiguous discrimination* [28]. Not all sets of pure states allow for unambiguous discrimination; a necessary and sufficient condition for unambiguous discrimination of a set of pure states is that the states should be linearly independent [16].

The objective is to find a measurement that minimizes the probability of the inconclusive outcome

$$p_?(\mathcal{E}_\chi, E) = \sum_{i=1}^N \eta_i \,\mathrm{Tr}\left(E_0 \left|\chi_i\middle\rangle\middle\langle\chi_i\right|\right). \tag{28}$$

Thus, the the maximum probability of conclusively identifying the state is given by

$$p^{ud}(\mathcal{E}_\chi) = 1 - \min_E p_?(\mathcal{E}_\chi, E) = \max_{\{p_i\}} \sum_{i=1}^N \eta_i p_i. \tag{29}$$

Finding the optimal probability of success is hard for a general problem, with solutions being known only for the two-state case [29] and in some cases involving symmetries and constraints [30, 31]. However, it can be cast as a semidefinite program [32, 33]

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^N \eta_i p_i \\ \text{subject to} \quad & \Gamma - P \succeq 0 \\ & P \succeq 0 \end{aligned} \tag{30}$$

where $P = \mathrm{diag}(p_1, \ldots, p_N)$ and Γ is the Gram matrix of the set of states having the entries $\Gamma_{ij} = \langle \chi_i | \chi_j \rangle$. The dual to the above problem is [34]

$$\begin{aligned} \underset{Z, \vec{z}}{\text{minimize}} \quad & \mathrm{Tr}(\Gamma Z) \\ \text{subject to} \quad & z_i + \eta_i - Z_{ii} = 0 \\ & Z, \vec{z} \succeq 0. \end{aligned} \tag{31}$$

where Z_{ii} denotes the i^{th} diagonal entry of Z.

4.1 Proof of LOCC optimality

The states of $\mathcal{T}$ admit unambiguous discrimination since they are linearly independent. We can lower bound the optimum probability of success by using the following result which provides the optimum probability for unambiguous discrimination of any set of equiprobable linearly independent states whose pairwise inner products are equal and real.

Lemma 11. *([19]) Let $S_N = \{|\psi_i\rangle : 2 \leqslant i \leqslant N\}$ be a set of equally likely, linearly independent pure states with the property $\langle \psi_i | \psi_j \rangle = s$ for $i \neq j$, where $s \in \left(-\frac{1}{N-1}, 1\right)$. Then the optimum probability for unambiguous discrimination is*

$$p = \begin{cases} 1 - s, & s \in [0, 1) \\ 1 + (N-1)\, s, & s \in \left(-\frac{1}{N-1}, 0\right]. \end{cases} \tag{32}$$

Using the above result, we can find a local protocol that unambiguously discriminates the members of $\mathcal{E}_{\mathcal{T}}$ with probability $(1-s)^2$. Given a state from $\mathcal{E}_{\mathcal{T}}$, one can successfully identify the state of the first component with probability $1 - s$. This leaves the second system in one of the two remaining states of $\mathcal{S}$ with equal probability, which again can be identified with probability $(1 - s)$. To conclusively identify the given state from $\mathcal{E}_{\mathcal{T}}$, one has to obtain conclusive outcomes for both measurements, and hence, the probability of success is $(1 - s)^2$. Note that this is a local protocol that uses only classical outcome of the first measurement to design the second measurement. Therefore,

$$p_G^{ud}(\mathcal{E}_{\mathcal{T}}) \geq (1 - s)^2. \tag{33}$$

We prove (16) by showing that this is the maximum probability of unambiguous discrimination allowed by quantum theory.

Lemma 12. *The optimal success probability of unambiguous discrimination of the elements of $\mathcal{E}_{\mathcal{T}}$ is given by $(1-s)^2$.*

Proof. We will consider the optimization problem in Equation (30) for different ensembles of states that are generated by permutations of the set $\{1, \ldots, 6\}$. First note that if σ is a permutation of $\{1, 2, \ldots, 6\}$, then the set of states

$$\mathcal{T}_\sigma = \left\{ \left| \psi_{\sigma(i)} \right\rangle \left| \psi_{\sigma(j)} \right\rangle \mid i, j \in \{1, 2, 3\} \text{ and } i \neq j \right\} \tag{34}$$

has the same Gram matrix as that of $\mathcal{T}$. Therefore, we can denote the Gram matrix of $\mathcal{T}_\sigma$ by Γ for any permutation σ. Moreover, if $P = \mathrm{diag}(p_1, \ldots, p_6)$ is the optimal solution for $\mathcal{E}_\mathcal{T}$, then the optimal solution for $\mathcal{E}_{\mathcal{T}_\sigma}$ is given by $P_\sigma = \mathrm{diag}\left(p_{\sigma(1)}, \ldots, p_{\sigma(6)}\right)$.

These solutions satisfy $\Gamma - P_\sigma \succeq 0$ for all permutations σ. These are 6! conditions, one for each permutation of the set $\{1, \ldots, 6\}$. We add them and use the fact that the sum of two positive semidefinite matrices is again positive semidefinite, to get

$$\Gamma - P_{\text{avg}} \succeq 0 \tag{35}$$

where

$$P_{\text{avg}} = \frac{1}{6!} \sum_\sigma P_\sigma = \lambda \mathbb{1} \tag{36}$$

and $\lambda = \left(\sum_{i=1}^6 p_i\right)/6$. This shows that optimal unambiguous discrimination of the members of $\mathcal{E}_\mathcal{T}$ is achieved by an equiprobable unambiguous discrimination [28]; therefore, the optimization problem can be expressed as

$$\begin{aligned} &\text{maximize} && \lambda \\ &\text{subject to} && \Gamma - \lambda \mathbb{1} \succeq 0 \\ & && \lambda \succeq 0. \end{aligned} \tag{37}$$

This is a standard SDP whose solution is given by the minimum eigenvalue of Γ [32] that corresponds to the optimum success probability. It is straightforward to compute the eigenvalues of Γ (this matrix is presented in Appendix C). The eigenvalues are

$$\left\{1 - s, 1 - s, (1 - s)^2, 1 + s - 2s^2, 1 + s - 2s^2, 1 + 2s + 3s^2\right\}. \tag{38}$$

It is easy to see that $(1 - s)^2 < 1 - s$. For the remaining two eigenvalues, observe that

$$\begin{aligned} 1 + s - 2s^2 &= (1 - s)^2 + 3s(1 - s) > (1 - s)^2, \text{ and} \\ 1 + 2s + 3s^2 &= (1 - s)^2 + 2s(2 + s) > (1 - s)^2. \end{aligned}$$

Thus $(1 - s)^2$ is the minimum eigenvalue of Γ, the Gram matrix of $\mathcal{T}$, over the entire interval $(0, 1)$. □

Since we have already established that there is a local protocol that succeeds in unambiguously discriminating the members of $\mathcal{E}_\mathcal{T}$ with probability $(1 - s)^2$, we have the following theorem

Theorem 13. *Optimal unambiguous discrimination of the members of $\mathcal{E}_\mathcal{T}$ is achievable by LOCC.*

It follows that $p_L^{ud}(\mathcal{E}_\mathcal{T}) = p_G^{ud}(\mathcal{E}_\mathcal{T})$, proving (16). This completes the proof of Theorem 1.

5 Conclusions

We have shown that quantum nonlocality without entanglement – a property ascribed to a set of product states that cannot be optimally discriminated by LOCC – is not always a property of the underlying states but a consequence of the chosen measure of state discrimination, i.e. a set of product states could be nonlocal under one measure but not for another. Specifically, we presented a family of sets of linearly independent product states, where each set has the following property: LOCC is sub-optimal for minimum-error discrimination of its members but optimal for unambiguous discrimination. The former is proved by showing that LOCC does not satisfy a condition that a measurement achieving the global optimum must; the latter is demonstrated by explicitly computing the global optimum and then showing that an LOCC protocol attains the same. In our examples the product states belong to $\mathbb{C}^3 \otimes \mathbb{C}^3$, so it would be interesting to see whether similar examples can be found in higher dimensions and multipartite systems.

References

[1] John S Bell. On the problem of hidden variables in quantum mechanics. *Reviews of Modern physics*, 38(3):447, 1966.

[2] Nicolas Brunner, Daniel Cavalcanti, Stefano Pironio, Valerio Scarani, and Stephanie Wehner. Bell nonlocality. *Reviews of modern physics*, 86(2):419, 2014.

[3] Charles H Bennett, David P DiVincenzo, Christopher A Fuchs, Tal Mor, Eric Rains, Peter W Shor, John A Smolin, and William K Wootters. Quantum nonlocality without entanglement. *Physical Review A*, 59(2):1070, 1999.

[4] János A Bergou, Ulrike Herzog, and Mark Hillery. Discrimination of quantum states. *Quantum state estimation*, pages 417–465, 2004.

[5] Anthony Chefles. Quantum state discrimination. *Contemporary Physics*, 41(6):401–424, 2000.

[6] Stephen M Barnett and Sarah Croke. Quantum state discrimination. *Advances in Optics and Photonics*, 1(2):238–278, 2009.

[7] Asher Peres and William K Wootters. Optimal detection of quantum information. *Physical Review Letters*, 66(9):1119, 1991.

[8] Somshubhro Bandyopadhyay and Michael Nathanson. Tight bounds on the distinguishability of quantum states under separable measurements. *Physical Review A*, 88(5):052313, 2013.

[9] Somshubhro Bandyopadhyay, Alessandro Cosentino, Nathaniel Johnston, Vincent Russo, John Watrous, and Nengkun Yu. Limitations on separable measurements by convex optimization. *IEEE Transactions on Information Theory*, 61(6):3593–3604, 2015.

[10] Eric Chitambar, Debbie Leung, Laura Mančinska, Maris Ozols, and Andreas Winter. Everything you always wanted to know about LOCC (but were afraid to ask). *Communications in Mathematical Physics*, 328:303–326, 2014.

[11] Alexander S Holevo. Statistical decision theory for quantum systems. *Journal of Multivariate Analysis*, 3(4):337–394, 1973.

[12] Igor D Ivanovic. How to differentiate between non-orthogonal states. *Physics Letters A*, 123(6):257–259, 1987.

[13] Dennis Dieks. Overlap and distinguishability of quantum states. *Physics Letters A*, 126(5-6):303–306, 1988.

[14] Asher Peres. How to differentiate between non-orthogonal states. *Physics Letters A*, 128(1-2):19, 1988.

[15] Joonwoo Bae. Structure of minimum-error quantum state discrimination. *New Journal of Physics*, 15(7):073037, 2013.

[16] Anthony Chefles. Unambiguous discrimination between linearly independent quantum states. *Physics Letters A*, 239(6):339–347, 1998.

[17] Eric Chitambar and Min-Hsiu Hsieh. Revisiting the optimal detection of quantum information. *Physical Review A—Atomic, Molecular, and Optical Physics*, 88(2):020302, 2013.

[18] Eric Chitambar, Runyao Duan, and Min-Hsiu Hsieh. When do local operations and classical communication suffice for two-qubit state discrimination? *IEEE Transactions on Information Theory*, 60(3):1549–1561, 2013.

[19] Luis Roa, Carla Hermann-Avigliano, R Salazar, and AB Klimov. Conclusive discrimination among n equidistant pure states. *Physical Review A*, 84(1):014302, 2011.

[20] Carlos Mochon. Family of generalized "pretty good" measurements and the minimal-error pure-state discrimination problems for which they are optimal. *Physical Review A—Atomic, Molecular, and Optical Physics*, 73(3):032328, 2006.

[21] Paul Hausladen and William K Wootters. A 'pretty good'measurement for distinguishing quantum states. *Journal of Modern Optics*, 41(12):2385–2390, 1994.

[22] Masahide Sasaki, Kentaro Kato, Masayuki Izutsu, and Osamu Hirota. Quantum channels showing superadditivity in classical capacity. *Physical Review A*, 58(1):146, 1998.

[23] Yonina C Eldar and G David Forney. On quantum detection and the square-root measurement. *IEEE Transactions on Information Theory*, 47(3):858–872, 2001.

[24] Nicola Dalla Pozza and Gianfranco Pierobon. Optimality of square-root measurements in quantum state discrimination. *Physical Review A*, 91(4):042334, 2015.

[25] David P DiVincenzo, Tal Mor, Peter W Shor, John A Smolin, and Barbara M Terhal. Unextendible product bases, uncompletable product bases and bound entanglement. *Communications in Mathematical Physics*, 238:379–410, 2003.

[26] Pratapaditya Bej and Saronath Halder. Unextendible product bases, bound entangled states, and the range criterion. *Physics Letters A*, 386:126992, 2021.

[27] Ping-Xing Chen and Cheng-Zu Li. Orthogonality and distinguishability: Criterion for local distinguishability of arbitrary orthogonal states. *Physical Review A*, 68(6):062107, 2003.

[28] DB Horoshko, MM Eskandari, and S Ya Kilin. Equiprobable unambiguous discrimination of quantum states by symmetric orthogonalisation. *Physics Letters A*, 383(15):1728–1732, 2019.

[29] Gregg Jaeger and Abner Shimony. Optimal distinction between two non-orthogonal quantum states. *Physics Letters A*, 197(2):83–87, 1995.

[30] H Sugimoto, T Hashimoto, M Horibe, and A Hayashi. Complete solution for unambiguous discrimination of three pure states with real inner products. *Physical Review A*, 82(3):032338, 2010.

[31] Yuqing Sun, Mark Hillery, and János A Bergou. Optimum unambiguous discrimination between linearly independent nonorthogonal quantum states and its optical realization. *Physical Review A*, 64(2):022311, 2001.

[32] Stephen P Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge University Press, 2004.

[33] Yonina C Eldar. A semidefinite programming approach to optimal unambiguous discrimination of quantum states. *IEEE Transactions on information theory*, 49(2):446–456, 2003.

[34] Tathagata Gupta, Shayeef Murshid, and Somshubhro Bandyopadhyay. Unambiguous discrimination of sequences of quantum states. *Physical Review A*, 109(5):052222, 2024.

[35] Sheldon Axler. *Linear algebra done right*. Springer, 2015.

Appendix

In this Appendix we present the proof of Lemma 8 and the Gram matrix of the states of $\mathcal{T}$ and its square root. Due to the symmetry of the problem in Lemma 8, two nonorthogonal ordered bases of $\mathbb{C}^3 \otimes \mathbb{C}^3$ lend themselves as natural bases to work in. These are (we suppress the $\otimes$ symbol to avoid clutter)

$$\begin{aligned}\mathcal{B} &= \{|\psi_1\rangle|\psi_2\rangle, |\psi_1\rangle|\psi_3\rangle, |\psi_2\rangle|\psi_1\rangle, |\psi_2\rangle|\psi_3\rangle, |\psi_3\rangle|\psi_1\rangle, |\psi_3\rangle|\psi_2\rangle, \{|\psi_i\rangle|\psi_i\rangle\}_{i=1}^3\}\\ \mathcal{B}' &= \{|\psi_1'\rangle|\psi_2'\rangle, |\psi_1'\rangle|\psi_3'\rangle, |\psi_2'\rangle|\psi_1'\rangle, |\psi_2'\rangle|\psi_3'\rangle, |\psi_3'\rangle|\psi_1'\rangle, |\psi_3'\rangle|\psi_2'\rangle, \{|\psi_i'\rangle|\psi_i'\rangle\}_{i=1}^3\},\end{aligned} \tag{39}$$

where $\left|\psi_i'\right\rangle$ is orthogonal to all vectors in $\mathcal{S}$ except $|\psi_i\rangle$, $i = 1, 2, 3$. Since these are nonorthogonal bases, we present a few elementary definitions from linear algebra in the following section, in order to review the methods of constructing matrix representation of operators in a nonorthogonal basis. All the material presented can be found in any standard linear algebra textbook such as [35].

A Representing vectors and operators in nonorthogonal basis

Let V be a vector space of dimension n. Let $\mathcal{B}_1 = \{|v_1\rangle, \ldots, |v_n\rangle\}$ be an ordered basis of V. If $|c\rangle \in V$ has the form

$$|c\rangle = \sum_{i=1}^{n} c_i |v_i\rangle \tag{40}$$

then the matrix representation of $|c\rangle$ *with respect to the basis* $\{|v_1\rangle, \ldots, |v_n\rangle\}$ is given by

$$|c\rangle = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}. \tag{41}$$

It is conventional to write $|c\rangle$ for both the vector and its matrix representation but we should bear in mind that the latter is basis dependent and people usually suppress the basis when it is understood from the context. A consequence of this definition is that, the basis vectors have a representation consisting of only 1 and 0 in their own basis,

$$|v_i\rangle = \begin{pmatrix} 0 & \ldots & 0 & \underbrace{1}_{i\text{-th position}} & 0 & \ldots & 0 \end{pmatrix}^T \tag{42}$$

where T denotes the transpose. Note that $\mathcal{B}_1$ need not be an orthogonal basis. This implies that in the Hilbert space $\mathbb{C}^2$, instead of the canonical basis of orthogonal states $|0\rangle$ and $|1\rangle$ if we use a basis consisting of nonorthogonal states, say $|0\rangle$ and $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, then these two will have the matrix representation

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad |+\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{43}$$

in their own basis, that is, $\{|0\rangle, |+\rangle\}$. Of course, in a nonorthogonal basis the inner product between two vectors

$$|a\rangle = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \qquad \text{and} \qquad |b\rangle = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \tag{44}$$

can no longer be computed by $\langle a|b\rangle = \sum_i a_i^* b_i$, a formula which holds only for orthogonal bases.

Let us now consider another vector space W of dimension m and consider a linear transformation $A : V \to W$. In addition to the basis $\mathcal{B}_1$ in V, fix another ordered basis $\mathcal{B}_2 =$

$\{|w_1\rangle, \dots, |w_n\rangle\}$ in W. Then the matrix of A *with respect to these bases* is the $m \times n$ matrix whose elements a_{jk} are defined by

$$A|v_k\rangle = a_{1k}|w_1\rangle + \cdots + a_{mk}|w_m\rangle. \tag{45}$$

Like vectors, it is conventional to denote both the operator and its matrix representation by A, but again, the latter is basis dependent which is usually implicitly understood. A consequence of this definition is that we lose the familiar matrix representation of operators in nonorthogonal bases. For example, in the basis $\{|0\rangle, |+\rangle\}$, the projection operator $|0\rangle\langle 0|$ has the representation

$$|0\rangle\langle 0| = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}. \tag{46}$$

Let us now consider two different ordered bases $\mathcal{B}_1 = \{|v_1\rangle, \dots, |v_n\rangle\}$ and $\mathcal{B}_2 = \{|v_1'\rangle, \dots, |v_n'\rangle\}$ of V, which are related as

$$|v_k\rangle = \sum_{i=1}^{n} a_{ik}|v_i'\rangle. \tag{47}$$

If $|c\rangle \in V$ can be written as

$$\begin{aligned} |c\rangle &= c_1|v_1\rangle + \cdots + c_n|v_n\rangle \\ &= c_1'|v_1'\rangle + \cdots + c_n'|v_n'\rangle \end{aligned} \tag{48}$$

then the coefficients are related by

$$\begin{pmatrix} c_1' \\ \vdots \\ c_n' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}. \tag{49}$$

Let us finish this section with one last remark. Suppose we have a Hermitian operator A acting on an n-dimensional vector space V. We have its matrix representation with respect to an arbitrary basis $\mathcal{B}_1$ and our task is to find the matrix representation of some function of that operator $f(A)$. We can find the matrix of $f(A)$ in the same basis $\mathcal{B}_1$ by the following procedure. First we solve the eigenvalue problem; the eigenvectors will be obtained in the basis $\mathcal{B}_1$. Let λ_i and $|e_i\rangle$ denote the i-th eigenvalue and its corresponding eigenvector respectively, and let the matrix of the k-th eigenvector be

$$|e_k\rangle = \begin{pmatrix} e_{k1} \\ \vdots \\ e_{kn} \end{pmatrix} \tag{50}$$

in the $\mathcal{B}_1$ basis. Construct the matrices P and D given by

$$P = \begin{pmatrix} |e_1\rangle & |e_2\rangle & \dots & |e_n\rangle \end{pmatrix} = \begin{pmatrix} e_{11} & e_{21} & \dots & e_{n1} \\ e_{12} & e_{22} & \dots & e_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ e_{1n} & e_{2n} & \dots & e_{nn} \end{pmatrix} \tag{51}$$

and

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}. \tag{52}$$

These matrices satisfy $AP = DP = PD$ from which it follows that

$$A = PDP^{-1}. \tag{53}$$

Note that the resultant matrix PDP^{-1} of the operator A is obtained in the same basis $\mathcal{B}_1$ that we started in, and this basis need not be orthogonal. To find the matrix of $f(A)$ in $\mathcal{B}_1$, we replace D by $f(D)$ whose i-th diagonal entry is $f(\lambda_i)$. Then we have $f(A) = Pf(D)P^{-1}$. Having reviewed all the necessary definitions, we now move on to the proof of Lemma 8.

B LOCC cannot distinguish the orthonormal basis $\tilde{\mathcal{B}}$

Proof. Suppose the states $\{|\mu_i\rangle\}$ are LOCC distinguishable. Then by Lemma 9, the product vectors $\{|\alpha_{ij}\rangle|\beta_{ij}\rangle\} \subset \mathbb{C}^3 \otimes \mathbb{C}^3$ that appear in the decomposition of $|\mu_i\rangle$ must belong to the 4-dimensional subspace orthogonal to the 5 vectors $\{|\mu_j\rangle\}_{j=1, j\neq i}^{6}$. A nonorthogonal basis for this subspace is given by $\left\{|\mu_i\rangle, \left|\psi_j'\right\rangle\left|\psi_j'\right\rangle\right\}_{j=1}^{3}$ where $|\psi_l'\rangle$ is orthogonal to all vectors in $\mathcal{S}$ except $|\psi_l\rangle$ and $l = 1, 2, 3$. Therefore, any product vector that appears in the decomposition of $|\mu_i\rangle$, say $|\alpha_{i1}\rangle|\beta_{i1}\rangle$, can be written as

$$|\alpha_{i1}\rangle|\beta_{i1}\rangle = c_0|\mu_i\rangle + c_1|\psi_1'\rangle|\psi_1'\rangle + c_2|\psi_2'\rangle|\psi_2'\rangle + c_3|\psi_3'\rangle|\psi_3'\rangle, \text{ where } c_0 \neq 0. \tag{54}$$

The condition of $c_0 \neq 0$ is necessary since $|\alpha_{i1}\rangle|\beta_{i1}\rangle$ must be nonorthogonal to $|\mu_i\rangle$ and $\left\{\left|\psi_j'\right\rangle\left|\psi_j'\right\rangle\right\}_{j=1}^{3}$ is orthogonal to the subspace $\mathcal{W}$ and hence to all $|\mu_i\rangle$'s. Since the $|\mu_i\rangle$'s are entangled, there must be at least two linearly independent product vectors $|\alpha_{i1}\rangle|\beta_{i1}\rangle$ and $|\alpha_{i2}\rangle|\beta_{i2}\rangle$ in the decomposition of $|\mu_i\rangle$ for all i. In what follows, we show that this is impossible.

To have notational convenience, let us, without loss of generality, consider two product vectors $|\alpha\rangle|\beta\rangle$ and $|\gamma\rangle|\delta\rangle$ that appears in the decomposition of, say, $|\mu_1\rangle$ and write them as

$$\begin{aligned} |\alpha\rangle|\beta\rangle &= a_0|\mu_1\rangle + a_1|\psi_1'\rangle|\psi_1'\rangle + a_2|\psi_2'\rangle|\psi_2'\rangle + a_3|\psi_3'\rangle|\psi_3'\rangle, \ a_0 \neq 0, \\ |\gamma\rangle|\delta\rangle &= b_0|\mu_1\rangle + b_1|\psi_1'\rangle|\psi_1'\rangle + b_2|\psi_2'\rangle|\psi_2'\rangle + b_3|\psi_3'\rangle|\psi_3'\rangle, \ b_0 \neq 0. \end{aligned} \tag{55}$$

Since $|\alpha\rangle|\beta\rangle$ is a product vector, we can expand each of $|\alpha\rangle$ and $|\beta\rangle$ in terms of the basis $\{|\psi_i'\rangle\}_{i=1}^{3}$ of $\mathbb{C}^3$,

$$\begin{aligned} |\alpha\rangle &= \alpha_1|\psi_1'\rangle + \alpha_2|\psi_2'\rangle + \alpha_3|\psi_3'\rangle \\ |\beta\rangle &= \beta_1|\psi_1'\rangle + \beta_2|\psi_2'\rangle + \beta_3|\psi_3'\rangle \end{aligned} \tag{56}$$

where $\alpha_i, \beta_i \in \mathbb{C}\ \forall i$. This gives us $|\alpha\rangle\,|\beta\rangle$ in the $\mathcal{B}'$ basis

$$\begin{aligned}|\alpha\rangle\,|\beta\rangle = {} & \alpha_1\beta_2\,|\psi_1'\rangle\,|\psi_2'\rangle + \alpha_1\beta_3\,|\psi_1'\rangle\,|\psi_3'\rangle + \alpha_2\beta_1\,|\psi_2'\rangle\,|\psi_1'\rangle + \alpha_2\beta_3\,|\psi_2'\rangle\,|\psi_3'\rangle + \\ & \alpha_3\beta_1\,|\psi_3'\rangle\,|\psi_1'\rangle + \alpha_3\beta_2\,|\psi_3'\rangle\,|\psi_2'\rangle + \alpha_1\beta_1\,|\psi_1'\rangle\,|\psi_1'\rangle + \alpha_2\beta_2\,|\psi_2'\rangle\,|\psi_2'\rangle + \alpha_3\beta_3\,|\psi_3'\rangle\,|\psi_3'\rangle\,.\end{aligned} \tag{57}$$

Inspecting the above equation, we see that for any product vector $|\zeta\rangle \in \mathbb{C}^3 \otimes \mathbb{C}^3$ whose components are $\{\zeta_i\}_{i=1}^9$ in the ordering of the $\mathcal{B}'$ basis given by (39), the following conditions must be satisfied

$$\begin{aligned}\zeta_1\zeta_9 &= \zeta_2\zeta_6 \\ \zeta_2\zeta_8 &= \zeta_1\zeta_4 \\ \zeta_4\zeta_7 &= \zeta_2\zeta_3.\end{aligned} \tag{58}$$

Let the components of $|\mu_1\rangle$ with respect to $\mathcal{B}'$ be denoted by $\{\mu_{1i}\}_{i=1}^9$. We can use Equation (42) to write $|\psi_i'\rangle\,|\psi_i'\rangle$ in the basis $\mathcal{B}'$,

$$|\psi_i'\rangle\,|\psi_i'\rangle = \begin{pmatrix}0 & \cdots & 0 & \underbrace{1}_{(6+i)\text{-th position}} & 0 & \cdots & 0\end{pmatrix}^T. \tag{59}$$

This allows us to express $|\alpha\rangle\,|\beta\rangle$ in $\mathcal{B}'$ basis using Equation (55) as

$$|\alpha\rangle\,|\beta\rangle_{\mathcal{B}'} = \begin{pmatrix}a_0\mu_{11} & a_0\mu_{12} & a_0\mu_{13} & a_0\mu_{14} & a_0\mu_{15} & a_0\mu_{16} & a_0\mu_{17}+a_1 & a_0\mu_{18}+a_2 & a_0\mu_{19}+a_3\end{pmatrix}^T. \tag{60}$$

Therefore, the conditions of Equation (58) for the state $|\alpha\rangle\,|\beta\rangle$ read

$$\begin{aligned}(a_0\mu_{11})(a_0\mu_{19}+a_3) &= a_0\mu_{12}\,a_0\mu_{16} \\ (a_0\mu_{12})(a_0\mu_{18}+a_2) &= a_0\mu_{11}\,a_0\mu_{14} \\ (a_0\mu_{14})(a_0\mu_{17}+a_1) &= a_0\mu_{12}\,a_0\mu_{13}\end{aligned} \tag{61}$$

which can be written as

$$a_i = k_i\,a_0,\ i = 1,2,3 \tag{62}$$

where

$$k_1 = \frac{\mu_{12}\mu_{13}}{\mu_{14}} - \mu_{17}, \quad k_2 = \frac{\mu_{11}\mu_{14}}{\mu_{12}} - \mu_{18} \text{ and } k_3 = \frac{\mu_{12}\mu_{16}}{\mu_{11}} - \mu_{19}. \tag{63}$$

Similarly, the conditions of Equation (58) for $|\gamma\rangle\,|\delta\rangle$ read

$$b_i = k_i\,b_0,\ i = 1,2,3. \tag{64}$$

We will soon show that for $s \in (0,1)$ the components μ_{11}, μ_{12} and μ_{14} do not equal 0, so that Equation (63) and hence Equations (62) and (64) are legitimate. For now let us consider Equations (62) and (64). These equations imply that the vectors $|\alpha\rangle\,|\beta\rangle$ and $|\gamma\rangle\,|\delta\rangle$ can be written as

$$\begin{aligned}|\alpha\rangle\,|\beta\rangle &= a_0\left(|\mu_1\rangle + k_1\,|\psi_1'\rangle\,|\psi_1'\rangle + k_2\,|\psi_2'\rangle\,|\psi_2'\rangle + k_3\,|\psi_3'\rangle\,|\psi_3'\rangle\right),\ a_0 \neq 0, \\ |\gamma\rangle\,|\delta\rangle &= b_0\left(|\mu_1\rangle + k_1\,|\psi_1'\rangle\,|\psi_1'\rangle + k_2\,|\psi_2'\rangle\,|\psi_2'\rangle + k_3\,|\psi_3'\rangle\,|\psi_3'\rangle\right),\ b_0 \neq 0\end{aligned} \tag{65}$$

which shows that they are linearly dependent. Therefore, any two product vectors in the span of $\{|\mu_i\rangle, \{\left|\psi'_j\right\rangle \left|\psi'_j\right\rangle\}_{j=1}^3\}$ with nonzero overlap with $|\mu_i\rangle$ are linearly dependent. This implies that the necessary conditions of Lemma 9 are not met, and as a result the states $\{|\mu_i\rangle\}$ cannot be distinguished by LOCC.

We now show that we are not dividing by zero in Equation (63) in the interval $s \in (0, 1)$, by equating μ_{11}, μ_{12} and μ_{14} to zero and solving for s. To this end, we first write the components of $|\mu_1\rangle$ in the $\mathcal{B}$ basis and then transform them to $\mathcal{B}'$. The components of $|\mu_1\rangle$ w.r.t. $\mathcal{B}$ can be obtained by noting that $|\mu_1\rangle = \rho^{-1/2} |\psi_1\rangle |\psi_2\rangle$, and consequently $|\mu_1\rangle$ is given by the first column of the matrix of $\rho^{-1/2}$ in $\mathcal{B}$. If $\{\lambda_i\}$ and $\{|\lambda_i\rangle\}$ are the eigenvalues and eigenvectors of ρ then $\rho^{-1/2}$ can be computed as

$$\rho^{-1/2} = \sum_i \frac{1}{\sqrt{\lambda_i}} |\lambda_i\rangle\langle\lambda_i|, \ \lambda_i \neq 0, \tag{66}$$

and the eigenvectors corresponding to the zero eigenvalues do not contribute to the sum in Equation (66). To write down ρ in $\mathcal{B}$, we note that

$$\rho = \frac{1}{6} \sum_{i=1}^{6} |\phi_i\rangle\langle\phi_i|. \tag{67}$$

Therefore the matrix of ρ in $\mathcal{B}$ can be found by its action on the basis vectors as given by Equation (45)

$$\begin{aligned} \rho \left|\phi_j\right\rangle &= \left(\frac{1}{6} \sum_{i=1}^{6} |\phi_i\rangle\langle\phi_i| \right) \left|\phi_j\right\rangle \\ &= \frac{1}{6} \sum_{i=1}^{6} \left\langle\phi_i|\phi_j\right\rangle |\phi_i\rangle \end{aligned} \tag{68}$$

where $j \in [9]$ and $\{|\phi_i\rangle\}_{i=7}^9$ are given by $\{|\psi_i\rangle |\psi_i\rangle\}_{i=1}^3$, respectively. This gives us

$$\rho_{\mathcal{B}} = \frac{1}{6} \begin{pmatrix} 1 & s & s^2 & s^2 & s^2 & s & s & s & s^2 \\ s & 1 & s^2 & s & s^2 & s^2 & s & s^2 & s \\ s^2 & s^2 & 1 & s & s & s^2 & s & s & s^2 \\ s^2 & s & s & 1 & s^2 & s^2 & s^2 & s & s \\ s^2 & s^2 & s & s^2 & 1 & s & s & s^2 & s \\ s & s^2 & s^2 & s^2 & s & 1 & s^2 & s & s \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{69}$$

From this we can compute $\rho^{-1/2}$ in $\mathcal{B}$ as

$$\rho_{\mathcal{B}}^{-1/2} = \frac{1}{\sqrt{6}} \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_3 & \gamma_1 & \gamma_4 & \gamma_4 & \gamma_5 \\ \gamma_1 & \gamma_0 & \gamma_3 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_4 \\ \gamma_2 & \gamma_3 & \gamma_0 & \gamma_1 & \gamma_3 & \gamma_1 & \gamma_4 & \gamma_4 & \gamma_5 \\ \gamma_3 & \gamma_1 & \gamma_1 & \gamma_0 & \gamma_3 & \gamma_2 & \gamma_5 & \gamma_4 & \gamma_4 \\ \gamma_3 & \gamma_2 & \gamma_3 & \gamma_3 & \gamma_0 & \gamma_1 & \gamma_4 & \gamma_5 & \gamma_4 \\ \gamma_1 & \gamma_3 & \gamma_1 & \gamma_2 & \gamma_1 & \gamma_0 & \gamma_5 & \gamma_4 & \gamma_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{70}$$

where

$$\gamma_0 = 2v_0 + 2v_1 + v_2 + v_3, \quad \gamma_1 = -v_0 + v_1 - v_2 + v_3, \quad \gamma_2 = 2v_0 - 2v_1 - v_2 + v_3,$$
$$\gamma_3 = -v_0 - v_1 + v_2 + v_3, \quad \gamma_4 = 2v_4 + 2v_5, \quad \gamma_5 = -4v_4 + 2v_5 \tag{71}$$

and

$$v_0 = \frac{1}{\sqrt{(1-s)}}, \quad v_1 = \frac{1}{\sqrt{-2s^2+s+1}}, \quad v_2 = \frac{1}{\sqrt{s^2-2s+1}},$$
$$v_3 = \frac{1}{\sqrt{3s^2+2s+1}}, \quad v_4 = \frac{s}{\sqrt{1-s}}, \quad v_5 = \frac{s^2+2s}{\sqrt{3s^2+2s+1}(9s^2+6s+3)}. \tag{72}$$

Reading off the first column of $\rho^{-1/2}$, we have

$$|\mu_1\rangle_{\mathcal{B}} = \frac{1}{\sqrt{6}} \begin{pmatrix} \frac{1}{\sqrt{3s^2+2s+1}} + \frac{2}{\sqrt{-2s^2+s+1}} + \frac{2}{\sqrt{1-s}} + \frac{1}{1-s} \\ \frac{1}{\sqrt{3s^2+2s+1}} + \frac{1}{\sqrt{-2s^2+s+1}} - \frac{1}{\sqrt{1-s}} + \frac{1}{s-1} \\ \frac{1}{\sqrt{3s^2+2s+1}} - \frac{2}{\sqrt{-2s^2+s+1}} + \frac{2}{\sqrt{1-s}} + \frac{1}{s-1} \\ \frac{1}{\sqrt{3s^2+2s+1}} - \frac{1}{\sqrt{-2s^2+s+1}} - \frac{1}{\sqrt{1-s}} + \frac{1}{1-s} \\ \frac{1}{\sqrt{3s^2+2s+1}} - \frac{1}{\sqrt{-2s^2+s+1}} - \frac{1}{\sqrt{1-s}} + \frac{1}{1-s} \\ \frac{1}{\sqrt{3s^2+2s+1}} + \frac{1}{\sqrt{-2s^2+s+1}} - \frac{1}{\sqrt{1-s}} + \frac{1}{s-1} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{73}$$

in the $\mathcal{B}$ basis.

To find $|\mu_1\rangle$ in $\mathcal{B}'$, we need to multiply the matrix of Equation (73) by the change-of-basis matrix M whose columns are the coefficients of the elements of $\mathcal{B}$ in $\mathcal{B}'$. That is, the (i,j)-th element of M will be the i-th coefficient of $|\phi_j\rangle$ in terms of $\mathcal{B}'$. This can be found out by first expressing the states of $\{|\psi_i\rangle\}$ in terms of $\{|\psi_i'\rangle\}$. By straightforward calculation we can find the relationship between $\mathcal{B}$ and $\mathcal{B}'$,

$$
\begin{aligned}
|\psi_i'\rangle &= \sqrt{\frac{1+s}{1+s-2s^2}}\left(|\psi_i\rangle - \frac{s}{1+s}\sum_{\substack{j=1\\ j\neq i}}^{3}|\psi_j\rangle\right), \\
|\psi_i\rangle &= \sqrt{\frac{1+s}{1+s-2s^2}}\left(|\psi_i'\rangle + s\sum_{\substack{j=1\\ j\neq i}}^{3}\left|\psi_j'\right\rangle\right), \quad i=1,2,3.
\end{aligned}
\tag{74}
$$

We now express the vectors of $\mathcal{B}$ in terms of the basis $\mathcal{B}'$ and construct the matrix M

$$
M = \frac{1+s}{1+s-2s^2}\begin{pmatrix}
1 & s & s^2 & s^2 & s^2 & s & s & s & s^2 \\
s & 1 & s^2 & s & s^2 & s^2 & s & s^2 & s \\
s^2 & s^2 & 1 & s & s & s^2 & s & s & s^2 \\
s^2 & s & s & 1 & s^2 & s^2 & s^2 & s & s \\
s^2 & s^2 & s & s^2 & 1 & s & s & s^2 & s \\
s & s^2 & s^2 & s^2 & s & 1 & s^2 & s & s \\
s & s & s & s^2 & s & s^2 & 1 & s^2 & s^2 \\
s & s^2 & s & s & s^2 & s & s^2 & 1 & s^2 \\
s^2 & s & s^2 & s & s & s & s^2 & s^2 & 1
\end{pmatrix}. \tag{75}
$$

We can now get $|\mu_1\rangle$ in $\mathcal{B}'$ by multiplying its matrix in $\mathcal{B}$ by M. This gives us

$$
|\mu_1\rangle_{\mathcal{B}'} = \sqrt{6}\begin{pmatrix}
-\frac{(s+1)\left(2\sqrt{-2s^2+s+1}-s+2\sqrt{1-s}+\sqrt{s(3s+2)+1}+1\right)}{6(s-1)(2s+1)} \\
\frac{(s+1)\left(-\sqrt{-2s^2+s+1}-s+\sqrt{1-s}-\sqrt{s(3s+2)+1}+1\right)}{6(s-1)(2s+1)} \\
-\frac{(s+1)\left(-2\sqrt{-2s^2+s+1}+s+2\sqrt{1-s}+\sqrt{s(3s+2)+1}-1\right)}{6(s-1)(2s+1)} \\
\frac{(s+1)\left(\sqrt{-2s^2+s+1}+s+\sqrt{1-s}-\sqrt{s(3s+2)+1}-1\right)}{6(s-1)(2s+1)} \\
\frac{(s+1)\left(\sqrt{-2s^2+s+1}+s+\sqrt{1-s}-\sqrt{s(3s+2)+1}-1\right)}{6(s-1)(2s+1)} \\
\frac{(s+1)\left(-\sqrt{-2s^2+s+1}-s+\sqrt{1-s}-\sqrt{s(3s+2)+1}+1\right)}{6(s-1)(2s+1)} \\
\frac{s(s+1)\left(2\sqrt{1-s}+s\left(\sqrt{1-s}-\sqrt{s(3s+2)+1}\right)+\sqrt{s(3s+2)+1}\right)}{3(1-s)^{3/2}(2s+1)\sqrt{s(3s+2)+1}} \\
\frac{s(s+1)\left(2\sqrt{1-s}+s\left(\sqrt{1-s}-\sqrt{s(3s+2)+1}\right)+\sqrt{s(3s+2)+1}\right)}{3(1-s)^{3/2}(2s+1)\sqrt{s(3s+2)+1}} \\
\frac{s(s+1)\left(s\sqrt{1-s}+2\sqrt{1-s}+2s\sqrt{s(3s+2)+1}-2\sqrt{s(3s+2)+1}\right)}{3(1-s)^{3/2}(2s+1)\sqrt{s(3s+2)+1}}
\end{pmatrix} \tag{76}
$$

in the basis $\mathcal{B}'$. To find the values of s for which μ_{11}, μ_{12} and μ_{14} equal zero, we set

$$
\mu_{11} = -\frac{(s+1)\left(2\sqrt{-2s^2+s+1}-s+2\sqrt{1-s}+\sqrt{s(3s+2)+1}+1\right)}{\sqrt{6}(s-1)(2s+1)} = 0 \tag{77a}
$$

$$\mu_{12} = \frac{(s+1)\left(-\sqrt{-2s^2+s+1} - s + \sqrt{1-s} - \sqrt{s(3s+2)+1} + 1\right)}{\sqrt{6}(s-1)(2s+1)} = 0 \tag{77b}$$

$$\mu_{14} = \frac{(s+1)\left(\sqrt{-2s^2+s+1} + s + \sqrt{1-s} - \sqrt{s(3s+2)+1} - 1\right)}{\sqrt{6}(s-1)(2s+1)} = 0. \tag{77c}$$

Solving these three equations, we get as solution $s = -1$ for Equation (77a) and $s = -1, 0$ for Equations (77b) and (77c). This completes our proof. □

C Gram matrix and square root calculations

The Gram matrix Γ of the set of states $\mathcal{T}$ is given by

$$\Gamma = \begin{pmatrix} 1 & s & s^2 & s^2 & s^2 & s \\ s & 1 & s^2 & s & s^2 & s^2 \\ s^2 & s^2 & 1 & s & s & s^2 \\ s^2 & s & s & 1 & s^2 & s^2 \\ s^2 & s^2 & s & s^2 & 1 & s \\ s & s^2 & s^2 & s^2 & s & 1 \end{pmatrix}. \tag{78}$$

Its square root is

$$\sqrt{\Gamma} = \frac{1}{6}\begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_3 & \gamma_1 \\ \gamma_1 & \gamma_0 & \gamma_3 & \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_0 & \gamma_1 & \gamma_1 & \gamma_3 \\ \gamma_3 & \gamma_1 & \gamma_1 & \gamma_0 & \gamma_3 & \gamma_2 \\ \gamma_3 & \gamma_2 & \gamma_1 & \gamma_3 & \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_3 & \gamma_3 & \gamma_2 & \gamma_1 & \gamma_0 \end{pmatrix} \tag{79}$$

where

$$\begin{gathered} \gamma_0 = 2v_0 + 2v_1 + v_2 + v_3, \quad \gamma_1 = v_1 - v_0 - v_2 + v_3, \\ \gamma_2 = -2v_1 + 2v_0 - v_2 + v_3, \quad \gamma_3 = -v_1 - v_0 + v_2 + v_3 \end{gathered} \tag{80}$$

and

$$v_0 = \sqrt{1-s}, \quad v_1 = \sqrt{-2s^2+s+1}, \quad v_2 = \sqrt{(s-1)^2}, \quad v_3 = \sqrt{3s^2+2s+1}. \tag{81}$$

Note that all the diagonal entries of $\sqrt{\Gamma}$ are equal.