Limitations on separable measurements by convex optimization

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State discrimination

The problem is specified by an ensemble

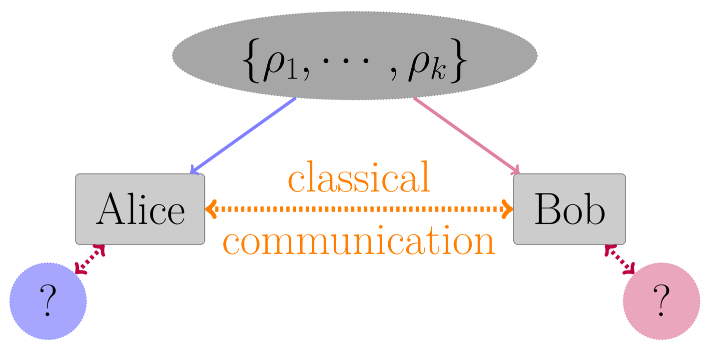
$$\{(p_1, \rho_1), \ldots, (p_N, \rho_N)\},$$

where

- N is a positive integer;
- $(p_1, \ldots, p_N)$ is the probability vector;
- $\rho_1, \ldots, \rho_N \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y})$ are density operators.

Problem

With respect to the probability vector $(p_1, \ldots, p_N)$, an index $k \in \{1, \ldots, N\}$ is selected at random, and Alice and Bob are given the quantum state ρ_k for the selected index k. Their goal is to determine the index k of the given state ρ_k by means of Local quantum Operations and Classical Communication (LOCC).



Typical assumptions:

- States are orthogonal
- Focus on perfect distinguishability
- Drawn from a uniform probability distribution

Example

- The four Bell states form a locally indistinguishable set.

$$|\psi_0\rangle = |0\rangle_\mathcal{X} |0\rangle_\mathcal{Y} + |1\rangle_\mathcal{X} |1\rangle_\mathcal{Y}$$
$$|\psi_1\rangle = |0\rangle_\mathcal{X} |0\rangle_\mathcal{Y} - |1\rangle_\mathcal{X} |1\rangle_\mathcal{Y}$$
$$|\psi_2\rangle = |0\rangle_\mathcal{X} |1\rangle_\mathcal{Y} + |1\rangle_\mathcal{X} |0\rangle_\mathcal{Y}$$
$$|\psi_3\rangle = |0\rangle_\mathcal{X} |1\rangle_\mathcal{Y} - |1\rangle_\mathcal{X} |0\rangle_\mathcal{Y}$$

- Optimal probability of distinguishing them is 1/2. [1]

PPT and separable measurements

Partial transpose $X \in \mathrm{L}(\mathcal{X} \otimes \mathcal{Y})$, $\mathrm{T}_\mathcal{X}(X) = (\mathrm{T} \otimes \mathbb{1}_\mathcal{Y})(X)$

Positive-partial-transpose operator $P \geq 0$ such that $\mathrm{T}_\mathcal{X}(P) \geq 0$ (symmetric w.r.t. $\mathcal{X}$ and $\mathcal{Y}$)

Separable operator

$$P = \sum_{k=1}^{M} Q_k \otimes R_k \in \mathrm{Sep}(\mathcal{X} : \mathcal{Y}),$$

for some choice of a positive integer M and positive semidefinite operators $Q_1, \ldots, Q_M \in \mathrm{Pos}(\mathcal{X})$ and $R_1, \ldots, R_M \in \mathrm{Pos}(\mathcal{Y})$;

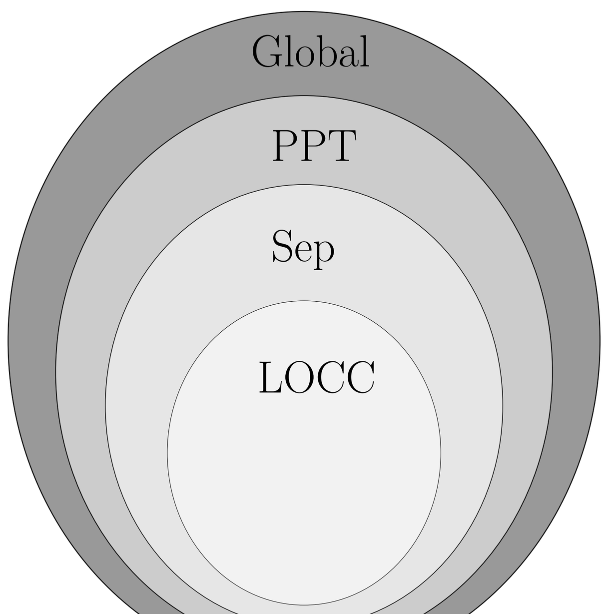
Separable (or PPT) measurement

$$\{P_1, \ldots, P_N\}$$

such that

- $P_1 + \cdots + P_N = \mathbb{1}$
- each $P_k \in \mathrm{Sep}(\mathcal{X} : \mathcal{Y})$ (or $P_k \in \mathrm{PPT}(\mathcal{X} : \mathcal{Y})$, respectively)

Classes of Measurements



LOCC – mathematically difficult object to characterize

Separable – optimizing over this set is NP-hard

PPT – efficient optimization via SDP

Cone Programming

- A generalization of linear and semidefinite programming

Primal problem (opt value α)

$$\begin{aligned} \text{maximize:} \quad & \langle A, X \rangle \\ \text{subject to:} \quad & \Phi(X) = B, \\ & X \in \mathcal{K}. \end{aligned}$$

Dual problem (opt value β)

$$\begin{aligned} \text{minimize:} \quad & \langle B, Y \rangle \\ \text{subject to:} \quad & \Phi^*(Y) - A \in \mathcal{K}^*, \\ & Y \in \mathrm{Herm}(\mathcal{W}). \end{aligned}$$

- Dual cone is defined as follows:

$$\mathcal{K}^* = \{Y \in \mathrm{Herm}(\mathcal{Z}) : \langle X, Y \rangle \geq 0 \text{ for all } X \in \mathcal{K}\}$$

- *Linear programming*: $\mathcal{K}$ is the positive orthant of $\mathbb{R}^n$
- *Semidefinite programming*: $\mathcal{K} = \mathrm{Pos}(\mathbb{C}^n)$, the cone of positive semidefinte operators.

Weak Duality Theorem: For every cone program, $\alpha \leq \beta$.

Optimizing over separable measurements

The maximum probability of distinguishing a set of states $\{\rho_1, \cdots, \rho_N\}$ by separable measurements can be expressed as the optimal value of a cone program:

Primal problem

$$\begin{aligned} \text{maximize:} \quad & p_1 \langle \rho_1, P_1 \rangle + \cdots + p_N \langle \rho_N, P_N \rangle \\ \text{subject to:} \quad & P_1 + \cdots + P_N = \mathbb{1}_{\mathcal{X} \otimes \mathcal{Y}} \\ & P_k \in \mathrm{Sep}(\mathcal{X} : \mathcal{Y}), \\ & \quad \text{for each } k = 1, \ldots, N. \end{aligned}$$

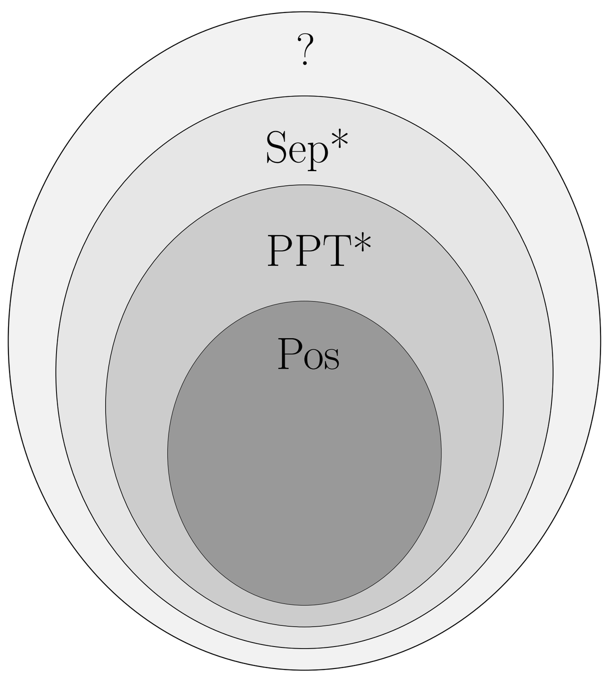
Dual problem

$$\begin{aligned} \text{minimize:} \quad & \mathrm{Tr}(H) \\ \text{subject to:} \quad & H - p_k \rho_k \in \mathrm{Sep}^*(\mathcal{X} : \mathcal{Y}), \\ & \quad \text{for each } k = 1, \ldots, N, \\ & H \in \mathrm{Herm}(\mathcal{X} \otimes \mathcal{Y}). \end{aligned}$$

Block-positive operators

Def. 1 $\mathrm{Sep}^*(\mathcal{X} : \mathcal{Y}) =$
$$\{H \in \mathrm{Herm}(\mathcal{X} \otimes \mathcal{Y}) : (x^* \otimes y^*) H (x \otimes y) \geq 0 \; \forall x \in \mathcal{X}, y \in \mathcal{Y}\}.$$

Def. 2 Choi representations of *positive* linear maps



Cone programming formulation

Analytically

- Many known families of positive linear maps
- Actual quantitative bounds

Computationally

- Optimizing over separable operators is NP-hard
- We can approximate the cone program by a hierarchy of SDPs based on symmetric extensions

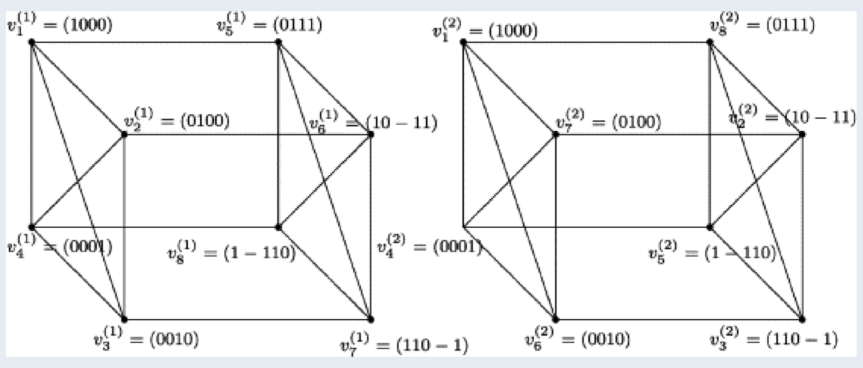
Applications

Unextendable Product Sets

Definition A set of mutually orthogonal product states s.t. it is impossible to find a nonzero product vector that is orthogonal to every element of the set.

Our results

- An easily checkable characterization of when an unextendable product set is perfectly discriminated by separable measurements.
- The first example of an unextendable product set that cannot be perfectly discriminated by separable measurements – in $\mathbb{C}^4 \otimes \mathbb{C}^4$.



- A proof that every unextendable product set together with one extra pure state orthogonal to every member of the unextendable product set is not perfectly discriminated by separable measurements.
- Related to a family of linear maps of Terhal.

Bell states

Our Result An optimal bound on the *entanglement cost* necessary to distinguish 3 or 4 Bell states.

- Related to a new family of positive linear maps.
- The lower bound is given by a standard teleportation protocol.

Yu–Duan–Ying states

Definition A set of 4 locally indistinguishable maximally entangled states in $\mathbb{C}^4 \otimes \mathbb{C}^4$ [4].

Our Result An optimal bound of 3/4 to distinguish the states by separable measurements.

- Connected to the positive maps of Breuer and Hall [5, 6].

Open Problems

- Entanglement cost to distinguish maximally entangled states in $\mathbb{C}^n \otimes \mathbb{C}^n$
- Are two copies sufficient to discriminate any set of orthogonal pure states?

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