Generalizations of Hedging Bets with Correlated Quantum Strategies

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Previous and Current Work

* Based on previous work "Hedging Bets with Correlated Quantum Strategies" - Molina, Watrous arXiv:1104.1140 [1].
* Joint work with Srinivasan Arunachalam, Abel Molina, and John Watrous.
The Protocol: Basic Setup

**Example**

**Game 1:**
- Alice: Pass
- Fail

**Game 2:**
- Alice: Pass
- Fail
Spoiler Alert

Classical Case
Optimal Probability:
- Passing both tests
  - $p^2$
- Passing at least one of the tests
  - $1 - (1 - p)^2$

Quantum Case
Optimal Probability:
- Passing both tests
  - $p^2$
- Passing at least one of the tests
  - $1$ (Spoiler)
**Spoiler Alert**

### Classical Case
**Optimal Probability:**
- Passing *both* tests
  - $p^2$
- Passing *at least one* of the tests
  - $1 - (1 - p)^2$

### Quantum Case
**Optimal Probability:**
- Passing *both* tests
  - $p^2$
- Passing *at least one* of the tests
  - 1 (Spoiler)
Formalization of the Testing Protocol (Running One Test)

Alice prepares: \( \rho \)
- \( u = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \)
- \( uu^* = \rho \in D (X \otimes Y) \)

Alice measures with respect to: \( \{ P_0, P_1 \} \)
- \( v = \cos(\theta) |00\rangle + \sin(\theta) |11\rangle \)
- \( vv^* = P_1 \in Pos (Y \otimes Z) \)
- \( \mathbb{I} - vv^* = P_0 \in Pos (Y \otimes Z) \)
Formalization of the Testing Protocol (Running One Test)

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Alice’s memory $\mathcal{Z}$
Formalization of the Testing Protocol (Running One Test)

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Alice measures with respect to: \( \{ P_0, P_1 \} \)
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- \( vv^* = P_1 \in \text{Pos}(Y \otimes Z), \quad I - vv^* = P_0 \in \text{Pos}(Y \otimes Z) \)
Bob’s goal is to optimize his probability of winning, which may be found by performing the inner product between the final state and a measurement operator $P_a \in \{P_0, P_1\}$.

The state $\sigma = (\Phi \otimes I_{L(Z)})(\rho)$ is the resulting state after Bob has applied his channel, $\Phi$, to the initial state $\rho$.

$$p(a) = \langle P_a, \sigma \rangle.$$
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Maximum and Minimum Measurement Probability

**Definition**

Maximum probability for outcome \( a \).

\[
M(a) = \max_{\Phi \in \mathcal{C}(\mathcal{X}, \mathcal{Y})} \langle P_a, (\Phi \otimes I_{\mathcal{L}(Z)})(\rho) \rangle
\]

**Definition**

Minimum probability for outcome \( a \).

\[
m(a) = \min_{\Phi \in \mathcal{C}(\mathcal{X}, \mathcal{Y})} \langle P_a, (\Phi \otimes I_{\mathcal{L}(Z)})(\rho) \rangle
\]

Note: max and min are used instead of sup and inf because they are being taken over a linear function on the compact set \( \mathcal{C}(\mathcal{X}, \mathcal{Y}) \).
Semidefinite Programs for $M(a)$ and $m(a)$

SDP for $M(a)$:  

Primal problem:  
maximize: $\langle Q_a, X \rangle$  
subject to: $\text{Tr}_Y(X) = \mathbb{I}_X$,  
$X \in \text{Pos}(Y \otimes X)$.

Dual problem:  
minimize: $\text{Tr}(Y)$  
subject to: $\mathbb{I}_Y \otimes Y \geq Q_a$,  
$Y \in \text{Herm}(X)$.

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Running One Test: Example

Running a specific single instance of the test

Alice → Bob
X

Z

Alice → Alice
Y

1 Pass

0 Fail
Step 1: Alice prepares her state

Alice prepares a pair of qubits in the state

$$u = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

and sends one qubit to Bob.
Step 2: Bob applies his channel:

In this instance, Bob’s best strategy is for $\Phi = \mathbb{I}$ since no matter what he does, it will not have an overall effect on the returned state since

$$\text{Tr}_Y(\sigma) = \text{Tr}_X(\rho) = \frac{1}{2} \mathbb{I}_Z.$$
Step 3: Alice measures

Alice measures with respect to $P_1 = vv^*$ and $P_0 = I - vv^*$ where

$$v = \cos(\pi/8) |00\rangle + \sin(\pi/8) |11\rangle$$
Running One Test: Bob’s Probability of Winning

Therefore, Bob’s maximum probability of winning is:

\[ M(1) = \langle P_1, \sigma \rangle = \cos^2(\pi/8) \approx 0.85. \]

And Bob’s minimum probability of losing is:

\[ m(0) = \langle P_0, \sigma \rangle = \sin^2(\pi/8) \approx 0.15. \]

Bob’s optimal channel in this case is defined as

\[ \Phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]
There exists a correlated *quantum* strategy for Bob where he will pass *at least one* of the tests with *certainty*.
Step 1: Alice prepares her state

The state over both games in terms of $u$ is defined as

$$u \otimes u = \frac{1}{2} (|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)$$
Step 2: Bob applies his channel

It can be shown (by running the SDP) that

$$\frac{1}{2}(-|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)$$
Step 3: Alice measures

Selecting $w = -\sin(\pi/8)|00\rangle + \cos(\pi/8)|11\rangle$ orthogonal to $v$, we can write Bob’s returned state as

$$-\frac{1}{2}|0000\rangle + \frac{1}{2}|0011\rangle + \frac{1}{2}|1100\rangle + \frac{1}{2}|1111\rangle = \frac{1}{\sqrt{2}}v \otimes w + \frac{1}{\sqrt{2}}w \otimes v$$
Step 3: Alice measures

In other words, the final state is written in terms of a superposition of either passing the first test and failing the second or failing the first test and passing the second.

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} v \\ w \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} w \\ v \end{pmatrix}
\]

Bob wins the first, and loses the second  
Bob loses the first, and wins the second
We saw a specific instance of improvement in Bob’s probability when a quantum strategy is adopted for winning 1 out of 2 repetitions of the test.

Question: Can this behavior be generalized for winning $1/n$? In other words, can we find an angle $\theta$ and a strategy $\Phi$ for Bob, such that he can always win $1/n$ with certainty?
Generalizing the Angle for Winning $1/n$:

Recall that \( \{P_0, P_1\} \) are defined in terms of
\[
\nu = \cos(\theta) |00\rangle + \sin(\theta) |11\rangle.
\]

\[
(\cos(\theta) + \sin(\theta))^{\otimes n} = 0 \iff (\cos(\theta) - \sin(\theta))^n = -2 \sin(\theta)^n \iff 
\cos(\theta) - \sin(\theta) = -(2^{1/n}) \sin(\theta) \iff \tan(x) = \frac{1}{1 - 2^{1/n}}
\]

The angles at which perfect hedging is achieved falls within the range:

\[
\theta \in \left[\tan^{-1}(2^{1/n} - 1), \tan^{-1}\left(\frac{1}{2^{1/n} - 1}\right)\right] \quad (1)
\]
Bob’s strategy for the end points of (1) can be characterized by:

\[ \Phi_1 = \sum_{i=0}^{n} (-1)^{AND(i) + PARITY(i)} |i\rangle \langle i|, \quad \Phi_2 = \sum_{i=0}^{n} (-1)^{OR(i) + PARITY(i)} |i\rangle \langle i| \]

In fact, Bob’s optimal strategy at any point is always unitary.
Connections to Other Areas of Research

- **Error reduction and the class QIP(2):**
  - Protocol was originally considered to study error reduction in QIP(2). The specific example for winning 1 out of 2 repetitions of the test illustrates that perfect parallel repetition cannot be used as a valid method of error reduction.

- **Quantum State Discrimination**
  - The hedging protocol can be viewed in the general sense as a framework for state discrimination.

- **Misc:**
  - Rank-1 One-Player Games
  - Cryptography
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Thank You!

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Questions? / Comments?


