# Generalizations of Hedging Bets with Correlated Quantum Strategies

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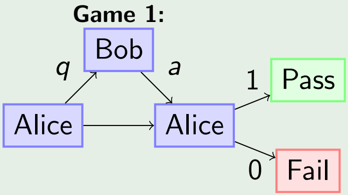
## Previous and Current Work

- Based on previous work "Hedging Bets with Correlated Quantum Strategies" - Molina, Watrous arXiv:1104.1140 [1].
- Joint work with Srinivasan Arunachalam, Abel Molina, and John Watrous.

# The Protocol: Basic Setup

## Example

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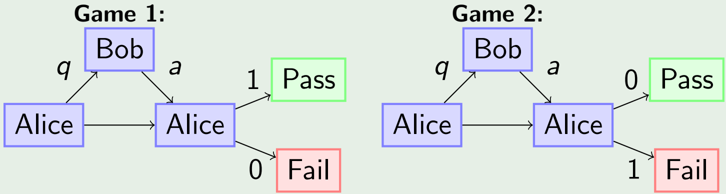

# Spoiler Alert

**Classical Case**
**Optimal Probability:**

- Passing *both* tests
  - $p^2$
- Passing *at least one* of the tests
  - $1 - (1 - p)^2$

**Quantum Case**
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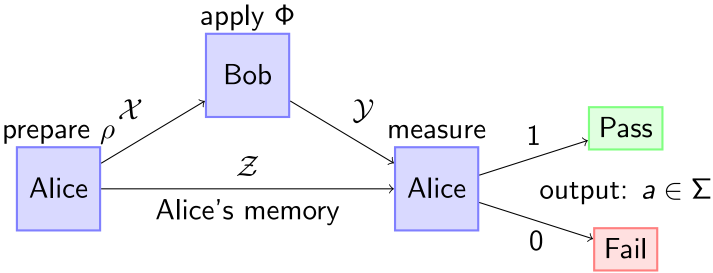
# Formalization of the Testing Protocol (Running One Test)




**Alice prepares:** $\rho$

- $u = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
  - $uu^* = \rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y})$

**Alice measures with respect to:** $\{P_0, P_1\}$

- $v = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle$
  - $vv^* = P_1 \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{Z}), \quad \mathbb{I} - vv^* = P_0 \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{Z})$

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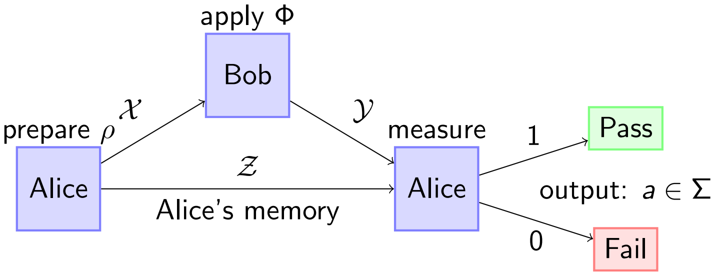


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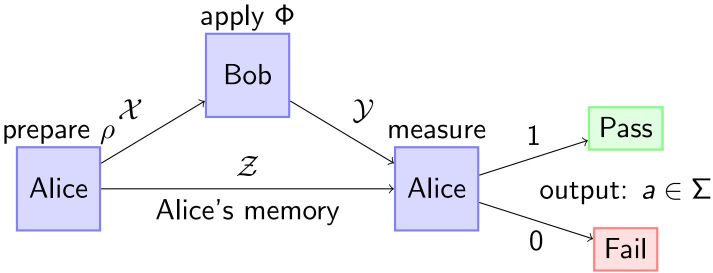


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## Bob's Goal

Bob's goal is to optimize his probability of winning, which may be found by performing the inner product between the final state and a measurement operator $P_a \in \{P_0, P_1\}$.

The state $\sigma = (\Phi \otimes \mathbb{I}_{\mathbf{L}(\mathcal{Z})})(\rho)$ is the resulting state after Bob has applied his channel, $\Phi$, to the initial state $\rho$.

$$p(a) = \langle P_a, \sigma \rangle.$$

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# Maximum and Minimum Measurement Probability

## Definition

Maximum probability for outcome $a$.

$$M(a) = \max_{\Phi \in \mathbf{C}(\mathcal{X},\mathcal{Y})} \langle P_a, (\Phi \otimes \mathbb{I}_{\mathbf{L}(\mathcal{Z})})(\rho) \rangle$$

## Definition

Minimum probability for outcome $a$.

$$m(a) = \min_{\Phi \in \mathbf{C}(\mathcal{X},\mathcal{Y})} \langle P_a, (\Phi \otimes \mathbb{I}_{\mathbf{L}(\mathcal{Z})})(\rho) \rangle$$

Note: max and min are used instead of sup and inf because they are being taken over a linear function on the compact set $\mathbf{C}(\mathcal{X}, \mathcal{Y})$.

# Semidefinite Programs for $M(a)$ and $m(a)$

**SDP for M(a):**

| Primal problem: | | Dual problem: | |
|---|---|---|---|
| maximize: | $\langle Q_a, X \rangle$ | minimize: | $\text{Tr}(Y)$ |
| subject to: | $\text{Tr}_{\mathcal{Y}}(X) = \mathbb{I}_{\mathcal{X}}$, | subject to: | $\mathbb{I}_{\mathcal{Y}} \otimes Y \geq Q_a$, |
| | $X \in \text{Pos}(\mathcal{Y} \otimes \mathcal{X})$. | | $Y \in \text{Herm}(\mathcal{X})$. |

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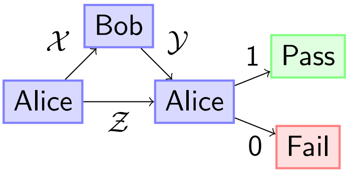
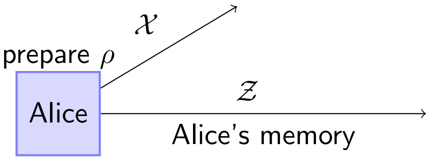
# Running One Test: Example

Running a specific single instance of the test



# Step 1: Alice prepares her state

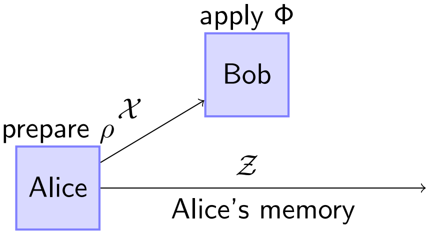



Alice prepares a pair of qubits in the state

$$u = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

and sends one qubit to Bob.

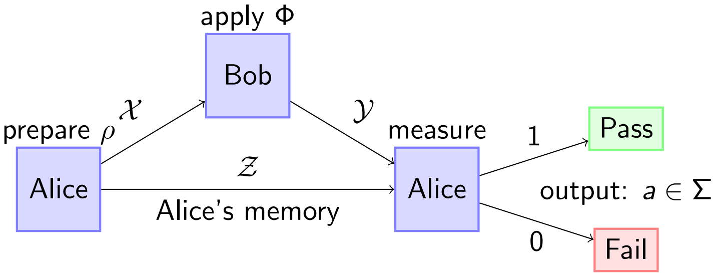
# Step 2: Bob applies his channel:




In this instance, Bob's best strategy is for $\Phi = \mathbb{I}$ since no matter what he does, it will not have an overall effect on the returned state since

$$\mathrm{Tr}_{\mathcal{Y}}(\sigma) = \mathrm{Tr}_{\mathcal{X}}(\rho) = \frac{1}{2}\mathbb{I}_{\mathcal{Z}}.$$

## Step 3: Alice measures




Alice measures with respect to $P_1 = vv^*$ and $P_0 = \mathbb{I} - vv^*$ where

$$v = \cos(\pi/8)\,|00\rangle + \sin(\pi/8)\,|11\rangle$$

# Running One Test: Bob's Probability of Winning

Therefore, Bob's maximum probability of winning is:

$$M(1) = \langle P_1, \sigma \rangle = \cos^2(\pi/8) \approx 0.85.$$

And Bob's minimum probability of losing is:

$$m(0) = \langle P_0, \sigma \rangle = \sin^2(\pi/8) \approx 0.15.$$

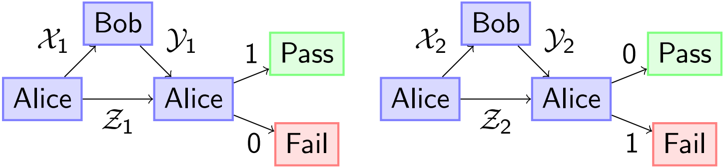
Bob's optimal channel in this case is defined as

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

# Running Two Tests: Example

There exists a correlated *quantum* strategy for Bob where he will pass *at least one* of the tests with *certainty*.

**Game 1:** **Game 2:**



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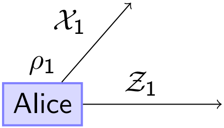
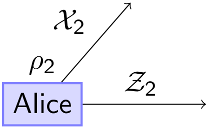
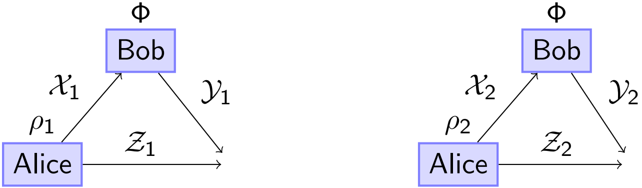



The state over both games in terms of $u$ is defined as

$$u \otimes u = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)$$

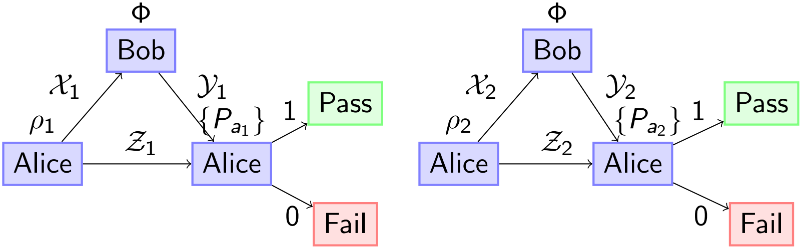
# Step 2: Bob applies his channel




It can be shown (by running the SDP) that

$$\frac{1}{2}(-|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)$$

# Step 3: Alice measures




Selecting $w = -\sin(\pi/8)|00\rangle + \cos(\pi/8)|11\rangle$ orthogonal to $v$, we can write Bob's returned state as

$$-\frac{1}{2}|0000\rangle + \frac{1}{2}|0011\rangle + \frac{1}{2}|1100\rangle + \frac{1}{2}|1111\rangle = \frac{1}{\sqrt{2}} v \otimes w + \frac{1}{\sqrt{2}} w \otimes v$$

# Step 3: Alice measures

In other words, the final state is written in terms of a superposition of either *passing the first test and failing the second* or *failing the first test and passing the second*.

$$\underbrace{\frac{1}{\sqrt{2}} v \otimes w}_{\text{Bob wins the first, and loses the second}} + \underbrace{\frac{1}{\sqrt{2}} w \otimes v}_{\text{Bob loses the first, and wins the second}}$$

## Generalizing This Behavior: Natural Question

- We saw a specific instance of improvement in Bob's probability when a quantum strategy is adopted for winning 1 out of 2 repetitions of the test.
  - **Question:** Can this behavior be generalized for winning $1/n$? In other words, can we find an angle $\theta$ and a strategy $\Phi$ for Bob, such that he can always win $1/n$ with certainty?

# Generalizing the Angle for Winning $1/n$:

Recall that $\{P_0, P_1\}$ are defined in terms of $v = \cos(\theta)\,|00\rangle + \sin(\theta)\,|11\rangle$.

$$(\cos(\theta) + \sin(\theta))^{\otimes n} = 0 \Leftrightarrow (\cos(\theta) - \sin(\theta))^n = -2\sin(\theta)^n \Leftrightarrow$$
$$\cos(\theta) - \sin(\theta) = -(2)^{1/n}\sin(\theta) \Leftrightarrow \tan(x) = \frac{1}{1 - 2^{1/n}}$$

The angles at which perfect hedging is achieved falls within the range:

$$\theta \in \left[\tan^{-1}\left(2^{1/n} - 1\right), \tan^{-1}\left(\frac{1}{2^{1/n} - 1}\right)\right] \tag{1}$$

# Generalizing the Strategy for Winning $1/n$

Bob's strategy for the end points of (1) can be characterized by:

$$\Phi_1 = \sum_{i=0}^{n} (-1)^{AND(i)+PARITY(i)} |i\rangle\langle i|, \quad \Phi_2 = \sum_{i=0}^{n} (-1)^{OR(i)+PARITY(i)} |i\rangle\langle i| \tag{2}$$

In fact, Bob's optimal strategy at any point is always unitary.

# Connections to Other Areas of Research

- Error reduction and the class QIP(2):
  - Protocol was originally considered to study error reduction in QIP(2). The specific example for winning 1 out of 2 repetitions of the test illustrates that perfect parallel repetition cannot be used as a valid method of error reduction.
- Quantum State Discrimination
  - The hedging protocol an be viewed in the general sense as a framework for state discrimination.
- Misc:
  - Rank-1 One-Player Games
  - Cryptography

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# Thank You!

Thanks!

Questions? / Comments?

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