

# ARKHIPOV'S THEOREM, GRAPH MINORS, AND LINEAR SYSTEM NONLOCAL GAMES

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ABSTRACT. The perfect quantum strategies of a linear system game correspond to certain representations of its solution group. We study the solution groups of graph incidence games, which are linear system games in which the underlying linear system is the incidence system of a (non-properly) two-coloured graph. While it is undecidable to determine whether a general linear system game has a perfect quantum strategy, for graph incidence games this problem is solved by Arkhipov's theorem, which states that the graph incidence game of a connected graph has a perfect quantum strategy if and only if it either has a perfect classical strategy, or the graph is nonplanar. Arkhipov's criterion can be rephrased as a forbidden minor condition on connected two-coloured graphs. We extend Arkhipov's theorem by showing that, for graph incidence games of connected two-coloured graphs, every quotient closed property of the solution group has a forbidden minor characterization. We rederive Arkhipov's theorem from the group theoretic point of view, and then find the forbidden minors for two new properties: finiteness and abelianness. Our methods are entirely combinatorial, and finding the forbidden minors for other quotient closed properties seems to be an interesting combinatorial problem.

## 1. INTRODUCTION

In quantum information, a nonlocal game is a type of cooperative game used to demonstrate the power of entanglement. Linear system nonlocal games are a class of nonlocal games in which the players try to prove to a referee that a linear system over a finite field (in this paper we restrict to  $\mathbb{Z}_2$ ) has a solution. Specifically, if  $Ax = b$  is a linear system over  $\mathbb{Z}_2$ , then the associated nonlocal game has a perfect deterministic strategy if and only if the system  $Ax = b$  has a solution. It was first observed by Mermin and Peres that there are linear systems  $Ax = b$  which do not have a solution, but where the associated game can be played perfectly if the players share an entangled quantum state [Mer90, Per91]. In general, whether or not the game associated to  $Ax = b$  has a perfect quantum strategy is controlled by a finitely presented group called the *solution group* of  $Ax = b$  [CM14, CLS17]. This group has a distinguished central generator  $J$  such that  $J^2 = 1$ , and the game has a perfect quantum strategy (resp. perfect commuting-operator strategy<sup>1</sup>) if and only if  $J$  is non-trivial in a finite-dimensional representation of the solution group (resp. is non-trivial in the solution group). It is shown in [Slo19b] that any finitely-presented group can be embedded in a solution group, so in this sense solution groups can be as complicated as arbitrary finitely-presented groups. In

<sup>1</sup>There are several different models of quantum strategies. *Quantum strategies* often refers to the most restrictive model of finite-dimensional strategies. *Commuting-operator strategies* refers to the most permissive model of infinite-dimensional strategies.

particular, it is undecidable to determine whether  $J$  is non-trivial in the solution group [Slo19b], and also undecidable to determine whether  $J$  is non-trivial in finite-dimensional representations of the solution group [Slo19a].

In this paper, we study the subclass of linear system games in which every variable of the linear system  $Ax = b$  appears in exactly two equations, or equivalently, in which each column of  $A$  contains exactly two non-zero entries. An  $n \times m$  matrix  $A$  satisfies this condition if and only if it is the incidence matrix  $\mathcal{I}(G)$  of a graph  $G$  with  $n$  vertices and  $m$  edges. After making this identification, the vector  $b \in \mathbb{Z}_2^n$  can be regarded as a function from the vertices of  $G$  to  $\mathbb{Z}_2$ , or in other words as a (not necessarily proper)  $\mathbb{Z}_2$ -colouring of  $G$ . Hence we refer to this subclass of linear system games as *graph incidence games*. Given a  $\mathbb{Z}_2$ -coloured graph  $(G, b)$ , we let  $\mathcal{G}(G, b)$  be the associated graph incidence game, and  $\Gamma(G, b)$  be the associated solution group, which we now call the *graph incidence group*. We let  $J_{G,b}$  denote the distinguished central element of  $\Gamma(G, b)$ , although we refer to this element as  $J$  if the  $\mathbb{Z}_2$ -coloured graph  $(G, b)$  is clear from context.

There is a simple criterion for whether a graph incidence game  $\mathcal{G}(G, b)$  has a perfect deterministic strategy: if  $G$  is a connected graph, then the linear system  $\mathcal{I}(G)x = b$  has a solution if and only if  $b$  has even parity, where the parity of a colouring is the sum  $\sum_{v \in V(G)} b(v)$  in  $\mathbb{Z}_2$ . If  $G$  is not connected, then  $\mathcal{I}(G)x = b$  has a solution if and only if the restriction of  $b$  to each connected component has even parity. If  $G$  is connected and  $b$  has odd parity, then  $\mathcal{G}(G, b)$  no longer has a perfect deterministic strategy, but there are still graphs  $G$  such that  $\mathcal{G}(G, b)$  has a perfect quantum strategy. In fact, Mermin and Peres' original examples of linear systems with perfect quantum strategies and no perfect deterministic strategies—the *magic square* and *magic pentagram* games—are examples of graph incidence games. The magic square game is  $\mathcal{G}(K_{3,3}, b)$ , where  $K_{r,s}$  is the complete bipartite graph with  $r$  vertices in one partition and  $s$  vertices in another, and the magic pentagram is  $\mathcal{G}(K_5, b)$ , where  $K_r$  is the complete graph on  $r$  vertices. Recall that Wagner's theorem famously states that a graph is nonplanar if and only if it contains  $K_{3,3}$  or  $K_5$  as a graph minor [Wag37]. The following theorem of Arkhipov shows that this connection between planarity and quantum strategies for  $\mathcal{G}(G, b)$  is not a coincidence:

**Theorem 1.1** (Arkhipov's theorem [Ark12]). *If  $G$  is a connected graph, then the graph incidence game  $\mathcal{G}(G, b)$  has a perfect quantum strategy if and only if either  $b$  has even parity, or  $b$  has odd parity and  $G$  is nonplanar.*

Another way to state Arkhipov's theorem is that if  $G$  is connected, then  $\mathcal{G}(G, b)$  has a perfect quantum strategy and no perfect classical strategy if and only if  $b$  has odd parity and  $G$  is nonplanar. The theorem also extends easily to disconnected graphs; however, to avoid complicating theorem statements, we focus on connected graphs in the introduction, and handle disconnected graphs later. Also, although it is not stated in [Ark12], the proof of Arkhipov's theorem implies that  $\mathcal{G}(G, b)$  has a perfect quantum strategy if and only if  $\mathcal{G}(G, b)$  has a perfect commuting-operator strategy.

Since graph planarity can be tested in linear time [HT74], Arkhipov's theorem implies that it is easy to tell if  $\mathcal{G}(G, b)$  has a perfect quantum strategy (or equivalently if  $J = 1$  in the graph incidence group  $\Gamma(G, b)$ ). This suggests that while there are interesting examples of graph incidence games, graph incidence groups are more tractable than the solution groups of general linear systems. Arkhipov's theorem

also suggests a connection between graph incidence groups and graph minors. The purpose of this paper is to develop these two points further. In particular we show that, for a natural extension of the notion of graph minors to  $\mathbb{Z}_2$ -coloured graphs, there is a strong connection between graph incidence groups and  $\mathbb{Z}_2$ -coloured graph minors:

**Lemma 1.2.** If  $(H, c)$  is a  $\mathbb{Z}_2$ -coloured graph minor of a  $\mathbb{Z}_2$ -coloured graph  $(G, b)$ , then there is a surjective group homomorphism  $\Gamma(G, b) \rightarrow \Gamma(H, c)$  sending  $J_{G,b} \mapsto J_{H,c}$ .

A *group over  $\mathbb{Z}_2$*  is a group  $\Phi$  with a distinguished central element  $J_\Phi$  such that  $J_\Phi^2 = 1$ , and a *morphism  $\Phi \rightarrow \Psi$  of groups over  $\mathbb{Z}_2$*  is a group homomorphism  $\Phi \rightarrow \Psi$  sending  $J_\Phi \rightarrow J_\Psi$  (this terminology comes from [Slo19a]). Although we won't use this statement, the proof of Lemma 1.2 implies that  $(G, b) \mapsto \Gamma(G, b)$  is a functor from the category of  $\mathbb{Z}_2$ -coloured graphs with  $\mathbb{Z}_2$ -coloured minor operations to the category of groups over  $\mathbb{Z}_2$  with surjective homomorphisms. It's also natural to define the graph incidence group  $\Gamma(G)$  of an ordinary graph  $G$  without the  $\mathbb{Z}_2$ -colouring, and this gives a functor from the category of graphs with the usual graph minor operations to the category of groups with surjective homomorphisms.

A property  $P$  of groups is *quotient closed* if for every surjective homomorphism  $\Phi \rightarrow \Psi$  between groups  $\Phi$  and  $\Psi$ , if  $P$  holds for  $\Phi$  then it also holds for  $\Psi$ . Similarly, we say that a property  $P$  of groups over  $\mathbb{Z}_2$  is *quotient closed* if for every surjective homomorphism  $\Phi \rightarrow \Psi$  of groups over  $\mathbb{Z}_2$ , if  $P$  holds for  $(\Phi, J_\Phi)$  then  $P$  holds for  $(\Psi, J_\Psi)$ . There are many well-known quotient closed properties of groups, including abelianness, finiteness, nilpotency, solvability, amenability, and Kazhdan's property (T). Any property of groups is also a property of groups over  $\mathbb{Z}_2$ , but properties of groups over  $\mathbb{Z}_2$  can also refer to the distinguished element  $J_\Phi$ . For instance, " $J_\Phi = 1$ " is a quotient closed property of groups over  $\mathbb{Z}_2$ , but does not make sense as a group property. If  $P$  is a quotient closed property of groups over  $\mathbb{Z}_2$ , then " $\Gamma(G, b)$  has property  $P$ " is a  $\mathbb{Z}_2$ -coloured minor-closed property of  $\mathbb{Z}_2$ -coloured graphs. The Robertson-Seymour theorem states that if  $P'$  is a minor-closed property of graphs, then there is a finite set  $\mathcal{F}$  of graphs such that  $G$  has  $P'$  if and only if  $G$  avoids  $\mathcal{F}$ , meaning that  $G$  does not contain any graphs in  $\mathcal{F}$  as a minor [RS04]. Using the Robertson-Seymour theorem and Lemma 1.2, it is possible to prove:

**Corollary 1.3.** If  $P$  is a quotient closed property of groups over  $\mathbb{Z}_2$ , then there is a finite set  $\mathcal{F}$  of  $\mathbb{Z}_2$ -coloured graphs such that if  $G$  is connected, then  $\Gamma(G, b)$  has  $P$  if and only if  $(G, b)$  avoids  $\mathcal{F}$ .

In particular, if  $P$  is quotient closed, then it is possible to check whether  $\Gamma(G, b)$  has  $P$  in time polynomial in the size of  $G$ . The set  $\mathcal{F}$  of graphs is called a set of *forbidden minors* for  $P$ .

The restriction to connected graphs in Corollary 1.3 is necessary when dealing with properties of groups over  $\mathbb{Z}_2$ , although for many natural properties (including  $J_\Phi = 1$ ), characterizing the connected case is enough to characterize the disconnected case. As with Lemma 1.2, there is also a version of Corollary 1.3 for ordinary graphs  $G$  and groups  $\Gamma(G)$ , and for this version the restriction to connected graphs is unnecessary. After giving more background on graph incidence games and groups in Section 2, the proofs of Lemma 1.2 and Corollary 1.3, along with the versions of these results for uncoloured graphs, are given in Section 3.

In the language of Corollary 1.3, Arkhipov’s theorem is equivalent to the statement that for a connected graph  $G$ ,  $J_{G,b} = 1$  if and only if  $(G, b)$  avoids  $(K_{3,3}, b)$  and  $(K_5, b)$  with  $b$  odd parity, and  $(K_1, b)$  with  $b$  even parity, where  $K_1$  is the single vertex graph. Corollary 1.3 explains why having a perfect quantum strategy can be characterized by avoiding a finite number of graphs. However, it doesn’t explain why, in the odd parity case, the minors are exactly the minors for planarity. An intuitive explanation for the connection with planarity is provided by the fact that relations in a group can be captured by certain planar graphs, called *pictures*. We reprove Arkhipov’s theorem using Lemma 1.2 and pictures in Section 4, and observe that Arkhipov’s theorem can be thought of as a stronger version of a result of Archdeacon and Richter that a graph is planar if and only if it has an odd planar cover [AR90]. Although our proof of Arkhipov’s theorem is phrased in a different language, at its core our proof is quite similar to the original proof, with the exception that our proof uses algebraic graph minors, while Arkhipov’s original proof uses topological graph minors.

Corollary 1.3 raises the question of whether we can find the forbidden graphs for other quotient closed properties of graph incidence groups. One particularly interesting property is finiteness. When  $\Gamma(G, b)$  is finite, perfect strategies for  $\mathcal{G}(G, b)$  are direct sums of irreducible representations of  $\Gamma(G, b)$  (see Corollary 2.10). Using the Gowers-Hatami stability theorem [GH17], Coladangelo and Stark [CS17] show that if  $\Gamma(G, b)$  is finite then  $\mathcal{G}(G, b)$  is robust, in the sense that, after applying local isometries, every almost-perfect strategy is close to a perfect strategy. Robustness is important in the study of self-testing and device-independent protocols in quantum information, see e.g. [MYS12, NV17, CGS17] for a small sample of results. Using Lemma 1.2, we give the following characterization of finiteness:

**Theorem 1.4.** The graph incidence group  $\Gamma(G, b)$  is finite if and only if  $G$  avoids  $C_2 \sqcup C_2$  and  $K_{3,6}$ .

Here  $C_2$  is the 2-cycle, i.e. a multigraph with 2 vertices and 2 edges between them, and  $C_2 \sqcup C_2$  is the graph with two connected components each isomorphic to  $C_2$ . A graph contains  $C_2 \sqcup C_2$  as a minor if and only if it has two vertex disjoint cycles. The characterization in Theorem 1.4 is in terms of the usual minors of  $G$ , rather than  $\mathbb{Z}_2$ -coloured minors of  $(G, b)$ , because finiteness of  $\Gamma(G, b)$  turns out to be independent of  $b$ .

We can also characterize when  $\Gamma(G, b)$  is abelian:

**Theorem 1.5.** If  $(G, b)$  is a  $\mathbb{Z}_2$ -coloured graph and  $G$  is connected, then  $\Gamma(G, b)$  is abelian if and only if  $(G, b)$  avoids the graphs  $(K_{3,3}, b')$  with  $b'$  odd parity,  $(K_5, b')$  with  $b'$  odd parity,  $(K_{3,4}, b')$  with  $b'$  even parity, and  $(C_2 \sqcup C_2, b')$  with  $b'$  any parity.

In terms of ordinary minors, if  $G$  is connected then  $\Gamma(G, b)$  is abelian if and only if either  $b$  is even and  $G$  avoids  $C_2 \sqcup C_2$  and  $K_{3,4}$ , or  $b$  is odd and  $G$  avoids  $K_{3,3}$ ,  $K_5$ , and  $C_2 \sqcup C_2$  (i.e.  $G$  is planar and does not have two disjoint cycles). If  $b$  is odd and  $G$  is planar, then  $\mathcal{G}(G, b)$  does not have any perfect strategies. However, when  $b$  is even the game  $\mathcal{G}(G, b)$  always has a perfect deterministic strategy. In this case, the group  $\Gamma(G, b)$  is abelian if and only if all perfect quantum strategies are direct sums of deterministic strategies on the support of their state (see Corollary 2.14).

The key to the proofs of both Theorems 1.4 and 1.5 is that  $\Gamma(C_2 \sqcup C_2) = \mathbb{Z}_2 * \mathbb{Z}_2$ , and hence is infinite and nonabelian. Thus, by Lemma 1.2, if  $G$  contains two disjoint cycles then  $\Gamma(G, b)$  must be infinite and nonabelian. We can then use a theorem of

Lovasz [Lov65] which characterizes graphs which do not have two disjoint cycles. The graph incidence groups of most of the graphs in this characterization do not have interesting structure, but an exception is the family of graphs  $K_{3,n}$ , which we analyze in Subsection 5.3. The games  $\mathcal{G}(K_{m,n}, b)$  have also recently been studied in [AW20, AW22] under the title of magic rectangle games, although our results do not seem to overlap. The proofs of Theorems 1.4 and 1.5 are given in Section 5. The implications of these results for graph incidence games is explained further in Subsection 2.4.

As mentioned above, there are many other quotient closed properties of groups, and finding the minors for these properties seems to be an interesting problem. We finish the paper in Section 6 with some remarks about these open problems.

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## 2. GRAPH INCIDENCE NONLOCAL GAMES AND GROUPS

In this section we define graph incidence nonlocal games, deterministic and quantum strategies for such games, and graph incidence groups. Most of the concepts in this section come from the theory of linear system games in [CM14, CLS17], although we elaborate a bit in Corollaries 2.10 and 2.14 to explain the implications of finiteness and abelianness of the graph incidence group for quantum strategies.

Let  $G$  be a graph with vertex set  $V(G)$  of size  $n$ , and edge set  $E(G)$  of size  $m$ . Throughout the paper, graphs are allowed to have multiple edges between vertices, but loops are not allowed. We say that  $G$  is simple if there is at most one edge between every pair of vertices. For a vertex  $v \in V(G)$ , we let  $E(v)$  denote the subset of edges  $e \in E(G)$  incident to  $v$ . Recall that the incidence matrix  $\mathcal{I}(G)$  of a graph  $G$  is the  $n \times m$  matrix whose  $(v, e)$ th entry is 1 whenever  $e \in E(v)$  and 0 otherwise. Let  $b : V(G) \rightarrow \mathbb{Z}_2$  be a (not necessarily proper) vertex  $\mathbb{Z}_2$ -colouring of  $G$ . A solution  $\hat{x}$  of the linear system  $\mathcal{I}(G)x = b$  is a function  $\hat{x} : E(G) \rightarrow \mathbb{Z}_2$  assigning a label  $\hat{x}(e) \in \mathbb{Z}_2$  to every  $e \in E(G)$ , such that  $\sum_{e \in E(v)} \hat{x}(e) = b(v)$  for every  $v \in V(G)$ . A graph where  $\mathcal{I}(G)x = b$  does not have a solution is shown in Figure 1.

The graph incidence game  $\mathcal{G}(G, b)$  associated to  $(G, b)$  is a two-player nonlocal game, in which the players try to demonstrate that they have a solution of  $\mathcal{I}(G)x = b$ . In the game, the referee sends a vertex  $v \in V(G)$  to the first player, and another vertex  $u \in V(G)$  to the second player. The players, who are unable to communicate once the game starts, reply to the referee with functions  $f : E(v) \rightarrow \mathbb{Z}_2$  and  $g : E(u) \rightarrow \mathbb{Z}_2$  respectively, which satisfy the equations

$$(2.1) \quad \sum_{e \in E(v)} f(e) = b(v) \quad \text{and} \quad \sum_{e \in E(u)} g(e) = b(u).$$

They win if and only if  $f(e) = g(e)$  for every  $e \in E(u) \cap E(v)$ . In other words, the players assign a value from  $\mathbb{Z}_2$  to each edge incident to their given vertex, and they win if the assigned values agree on all edges incident to both  $u$  and  $v$ . If  $u \neq v$ , then the assigned values must agree on all edges between  $u$  and  $v$ , while if  $u = v$  then to win the functions must satisfy  $f(e) = g(e)$  for all  $e \in E(v)$ .

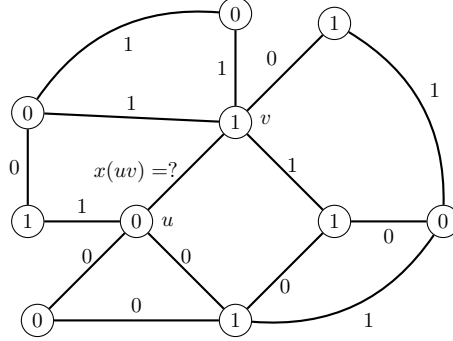


FIGURE 1. Assigning 0 or 1 to the edge  $uv$  would be inconsistent with the colouring of  $G$ , so there is no way to complete the given edge assignment to a solution of  $\mathcal{I}(G)x = b$ . In fact, for this graph  $\mathcal{I}(G)x = b$  has no solution.

**2.1. Perfect classical strategies for graph incidence games.** A strategy for a graph incidence game is said to be deterministic if each player's answer depends only on the vertex they receive. Formally, a deterministic strategy for a graph incidence game is specified by two collections of functions  $\{f_v : E(v) \rightarrow \mathbb{Z}_2\}_{v \in V(G)}$  and  $\{g_u : E(u) \rightarrow \mathbb{Z}_2\}_{u \in V(G)}$ , such that  $\sum_{e \in E(v)} f_v(e) = \sum_{e \in E(v)} g_u(e) = b(v)$  for all  $v \in V(G)$ . Such a strategy is *perfect* if the players win on every combination of inputs, i.e. if for every  $(v, u) \in V(G) \times V(G)$ ,  $f_v(e) = g_u(e)$  for all  $e \in E(v) \cap E(u)$ . In particular, if  $(\{f_v\}, \{g_u\})$  is perfect, then  $f_v = g_v$  for all  $v \in V(G)$ , and furthermore, if  $e$  is an edge with endpoints  $v$  and  $u$ , then  $f_v(e) = g_u(e) = f_u(e)$ . Thus, for any perfect deterministic strategy  $(\{f_v\}, \{g_u\})$ , there is a function  $\hat{x} : E(G) \rightarrow \mathbb{Z}_2$  such that  $f_v = g_v = \hat{x}|_{E(v)}$  for all  $v \in V(G)$ . Because  $\sum_{e \in E(v)} \hat{x}(e) = \sum_{e \in E(v)} f_v(e) = b(v)$  for every  $v \in V(G)$ ,  $\hat{x}$  is a solution to the linear system  $\mathcal{I}(G)x = b$ . Conversely, if  $\hat{x}$  is a solution to this linear system, then the strategy  $(\{f_v\}, \{g_u\})$  with  $f_v = g_v = \hat{x}|_{E(v)}$  is a perfect deterministic strategy, so there is a one-to-one correspondence between perfect deterministic strategies of  $\mathcal{G}(G, b)$  and solutions of the linear system  $\mathcal{I}(G)x = b$ .

It is not hard to see that when  $G$  is connected,  $\mathcal{I}(G)x = b$  has a solution if and only if the restriction of  $b$  to each connected component of  $G$  has even parity. If  $G$  is disconnected, then  $\mathcal{I}(G)$  is the direct sum of the incidence matrices for the connected components of  $G$ , and hence  $b$  is in the image of  $\mathcal{I}(G)$  if and only if the restriction of  $b$  to every connected component of  $G$  has even parity. We summarize these facts in the following lemma:

**Lemma 2.1.** Let  $(G, b)$  be a  $\mathbb{Z}_2$ -coloured graph, and let  $G_1, \dots, G_k$  be the connected components of  $G$ . The linear system  $\mathcal{I}(G)x = b$  has a solution (or equivalently,  $\mathcal{G}(G, b)$  has a perfect deterministic strategy) if and only if the restriction  $b|_{V(G_i)}$  has even parity for all  $1 \leq i \leq k$ .

A proof of Lemma 2.1 can be found in [Ark12].

Deterministic strategies belong to a larger class of strategies called *classical strategies*, which allow the players to use local and shared randomness. Every classical strategy is a convex combination of deterministic strategies, and hence a

nonlocal game has a perfect classical strategy if and only if it has a perfect deterministic strategy. Since we don't need the notion of a classical strategy, we omit the definition here.

**2.2. Perfect quantum strategies and the graph incidence group.** To define quantum strategies for graph incidence games, we require the following facts from quantum probability. A quantum state  $\nu$  is a unit vector in a complex Hilbert space  $\mathcal{H}$ . A  $\{\pm 1\}$ -valued observable  $\mathcal{O}$  is a self-adjoint unitary operator on  $\mathcal{H}$ , and should be thought of as a  $\{\pm 1\}$ -valued random variable. We let  $\mathcal{O}(\mathcal{H})$  denote the space of  $\{\pm 1\}$ -valued observables on  $\mathcal{H}$ . Given a quantum state  $\nu \in \mathcal{H}$  and an observable  $\mathcal{O} \in \mathcal{O}(\mathcal{H})$ , the expected value of the observable is given by the inner-product  $\langle \nu | \mathcal{O} \nu \rangle_{\mathcal{H}}$ . Two observables  $\mathcal{O}_A$  and  $\mathcal{O}_B$  are jointly measurable if they commute, in which case the product  $\mathcal{O}_A \mathcal{O}_B$  is the observable corresponding to the product of the values. Since  $\mathbb{Z}_2$  is isomorphic to the multiplicative group  $\{\pm 1\}$ , we can think of a  $\{\pm 1\}$ -valued observable as a  $\mathbb{Z}_2$ -valued observable, with value 1 corresponding to  $0 \in \mathbb{Z}_2$ , and value  $-1$  corresponding to  $1 \in \mathbb{Z}_2$ .

As mentioned in the introduction, there is more than one way to model quantum strategies for nonlocal games. The finite-dimensional model is the most restrictive, while the commuting-operator model is the most permissive. We call strategies in the former model “quantum strategies”, and strategies in the latter model “commuting-operator strategies”. Quantum strategies are a subset of commuting-operator strategies.

**Definition 2.2.** A *commuting-operator strategy* (resp. *quantum strategy*) for a graph incidence game  $\mathcal{G}(G, b)$  consists of a Hilbert space  $\mathcal{H}$  (resp. finite-dimensional Hilbert space  $\mathcal{H}$ ), a quantum state  $\nu \in \mathcal{H}$ , and two subsets

$$\{\mathcal{X}_{ve} : v \in V(G), e \in E(v)\} \text{ and } \{\mathcal{Y}_{ve} : v \in V(G), e \in E(v)\}$$

of  $\mathcal{O}(\mathcal{H})$  such that the pairs

$$\begin{aligned} &\mathcal{X}_{ve}, \mathcal{X}_{vf}, v \in V(G), e, f \in E(v), \\ &\mathcal{Y}_{ve}, \mathcal{Y}_{vf}, v \in V(G), e, f \in E(v), \text{ and} \\ &\mathcal{X}_{ve}, \mathcal{Y}_{uf}, v, u \in V(G), e \in E(v), f \in E(u) \end{aligned}$$

are all jointly measurable, and such that

$$(2.2) \quad \prod_{e \in E(v)} \mathcal{X}_{ve} = \prod_{e \in E(v)} \mathcal{Y}_{ve} = (-1)^{b(v)} \mathbb{1} \text{ for all } v \in V(G).$$

In the graph incidence game,  $\mathcal{X}_{ve}$  (resp.  $\mathcal{Y}_{ve}$ ) is the observable corresponding to the first (resp. second) player's assignment  $a$  to edge  $e$  upon receiving vertex  $v$ , where we identify the values  $\{\pm 1\}$  of the observables with  $\mathbb{Z}_2$  as above. The condition that the players' outputs satisfy Equation 2.1 is rewritten multiplicatively (due to the identification of  $\mathbb{Z}_2$  with  $\{\pm 1\}$ ) in Equation 2.2. In particular, if  $(\{f_v\}, \{g_u\})$  is a deterministic strategy for  $\mathcal{G}(G, b)$ , then the observables  $\mathcal{X}_{ve} = (-1)^{f_v(e)}$  and  $\mathcal{Y}_{uf} = (-1)^{g_u(f)}$  on  $\mathcal{H} = \mathbb{C}^1$  satisfy Equation 2.2, and conversely any one-dimensional quantum strategy must come from a deterministic strategy in this way.

To win  $\mathcal{G}(G, b)$ , the players' assignments to edge  $e$  should be perfectly correlated, and hence a commuting-operator strategy is *perfect* if the expectation  $\langle \nu | \mathcal{X}_{ve} \mathcal{Y}_{ue} \nu \rangle = 1$  for all  $v, u \in V(G)$  and  $e \in E(u) \cap E(v)$ . Again, perfect one-dimensional quantum strategies correspond to perfect deterministic strategies, and

hence to solutions of  $\mathcal{I}(G)x = b$ . Perfect commuting-operator strategies in higher dimensions can be understood using the following group:

**Definition 2.3.** Let  $(G, b)$  be a  $\mathbb{Z}_2$ -coloured graph. The *graph incidence group*  $\Gamma(G, b)$  is the group generated by  $\{x_e : e \in E(G)\} \cup \{J\}$  subject to the following relations:

- (1)  $J^2 = 1$  and  $x_e^2 = 1$  for all  $e \in E(G)$  (generators are involutions),
- (2)  $[x_e, J] = 1$  for all  $e \in E(G)$  ( $J$  is central),
- (3)  $[x_e, x_{e'}] = 1$  for all  $v \in V(G)$  and  $e, e' \in E(v)$  (edges incident to a vertex commute), and
- (4)  $\prod_{e \in E(v)} x_e = J^{b(v)}$  for all  $v \in V(G)$  (product of edges around a vertex is  $J^{b(v)}$ ).

Here  $[x, y] := xyx^{-1}y^{-1}$  is the commutator of  $x$  and  $y$ . Note that, after identifying  $\mathbb{Z}_2$  with  $\{\pm 1\}$ , the one-dimensional representations of  $\Gamma(G, b)$  with  $J \mapsto -1$  are the solutions of  $\mathcal{I}(G)x = b$ . Graph incidence groups are a special case of solution groups of linear systems (specifically,  $\Gamma(G, b)$  is the solution group of the linear system  $\mathcal{I}(G)x = b$ ). Solution groups were formally introduced in [CLS17], although the notion was essentially already present in [CM14]. Since it is not necessary to refer to the linear system  $\mathcal{I}(G)x = b$  when defining  $\mathcal{G}(G, b)$  and  $\Gamma(G, b)$ , we prefer the term graph incidence group in this context.

As mentioned in the introduction, the existence of perfect quantum strategies for  $\mathcal{G}(G, b)$  is connected to the graph incidence group in the following way:

**Theorem 2.4** ([CM14, CLS17]). The graph incidence game  $\mathcal{G}(G, b)$  has a perfect commuting-operator strategy if and only if  $J \neq 1$  in  $\Gamma(G, b)$ , and a perfect quantum strategy if and only if  $J$  is non-trivial in a finite-dimensional representation of  $\Gamma(G, b)$ .

Note that  $J \neq 1$  in  $\Gamma(G, b)$  if and only if  $J$  is non-trivial in some representation of  $\Gamma(G, b)$ . Also, since  $J$  is central and  $J^2 = 1$ , if  $J$  is non-trivial in a representation (resp. finite-dimensional representation) of  $\Gamma(G, b)$ , then there is a representation (resp. finite-dimensional representation) where  $J \mapsto -\mathbb{1}$ .

**2.3. Properties of graph incidence groups.** For context with Theorem 2.4, and for use in the next section, we explain some basic properties of graph incidence groups. First, we show that the isomorphism type of  $\Gamma(G, b)$  depends only on  $G$  and the parity of  $b$ .

**Lemma 2.5.** Let  $b$  and  $b'$  be  $\mathbb{Z}_2$ -colourings of a connected graph  $G$ . If  $b$  and  $b'$  have the same parity, then there is an isomorphism  $\Gamma(G, b) \cong \Gamma(G, b')$  sending  $J_{G,b} \mapsto J_{G,b'}$ .

*Proof.* If  $b$  and  $b'$  have the same parity, then  $\tilde{b} = b + b'$  has even parity, and the linear system  $\mathcal{I}(G)x = \tilde{b}$  has a solution  $\hat{x}$  by Lemma 2.1. It follows from Definition 2.3 that there is an isomorphism  $\Gamma(G, b) \rightarrow \Gamma(G, b')$  sending  $x_e \mapsto J^{\hat{x}(v)}x_e$  and  $J_{G,b} \mapsto J_{G,b'}$ .  $\square$

As stated in Lemma 2.1, whether  $\mathcal{G}(G, b)$  has a perfect deterministic strategy can be determined by looking at the connected components of  $G$ . Whether  $J = 1$  in  $\Gamma(G, b)$  can also be determined by looking at the connected components of  $G$ . Recall that the coproduct of a collection of group homomorphisms  $\psi_i : \Psi \rightarrow \Phi_i$ ,



$i = 1, \dots, k$ , is the quotient of the free product  $\Phi_1 * \dots * \Phi_k$  by the normal subgroup generated by the relations  $\psi_i(g) = \psi_j(g)$  for all  $1 \leq i, j \leq k$  and  $g \in \Psi$ . We denote this coproduct by  $\Psi \coprod_{i=1}^k \Phi_i$ . If each  $\psi_i$  is injective, so that  $\Psi$  is a subgroup of each  $\Phi_i$ , then  $\Psi \coprod_{i=1}^k \Phi_i$  is called the amalgamated free product of the groups  $\Phi_1, \dots, \Phi_k$  over  $\Psi$ , and it is well-known that the natural homomorphisms  $\Phi_i \rightarrow \Psi \coprod_{i=1}^k \Phi_i$  are also injective.

**Lemma 2.6.** If a  $\mathbb{Z}_2$ -coloured graph  $(G, b)$  has connected components  $G_1, \dots, G_k$ , and  $b_i$  is the restriction of  $b$  to  $G_i$ , then

$$\Gamma(G, b) = \langle J \rangle \coprod_{i=1}^k \Gamma(G_i, b_i),$$

where the coproduct is over the homomorphisms  $\mathbb{Z}_2 = \langle J \rangle \rightarrow \Gamma(G_i, b_i)$  sending  $J \mapsto J_{G_i, b_i}$ . In particular,  $J_{G, b} \neq 1$  in  $\Gamma(G, b)$  if and only if  $J_{G_i, b_i} \neq 1$  in  $\Gamma(G_i, b_i)$  for all  $i = 1, \dots, k$ , and if  $J_{G, b} \neq 1$  then the inclusions  $\Gamma(G_i, b_i) \rightarrow \Gamma(G, b)$  are injective.

*Proof.* The relations for  $\Gamma(G, b)$  can be grouped by component. The variable  $J$  is the only common generator between relations in different components, so the lemma follows directly from the presentation.  $\square$

The  $\mathbb{Z}_2$ -colouring  $b$  is used in the definition of  $\Gamma(G, b)$  to determine when to include the generator  $J$  in a relation, but if we replace  $J$  with the identity, then we get a similar group which only depends on the uncoloured graph  $G$ :

**Definition 2.7.** Let  $G$  be a graph. The *graph incidence group*  $\Gamma(G)$  is the finitely presented group with generators  $\{x_e : e \in E(G)\}$ , and relations (1)-(4) from Definition 2.3, with  $J$  replaced by 1.

Note that  $\Gamma(G) = \Gamma(G, b) / \langle J \rangle$  for any  $\mathbb{Z}_2$ -colouring  $b$  of  $G$ . For this reason,  $\Gamma(G)$  is finite if and only if  $\Gamma(G, b)$  is finite for some (resp. any)  $\mathbb{Z}_2$ -colouring  $b$ . Also, since  $J$  does not appear in any of the relations for  $\Gamma(G, 0)$ , we have  $\Gamma(G, 0) = \Gamma(G) \times \mathbb{Z}_2$ . By Lemmas 2.5 and 2.6, if the restriction of  $b$  to every connected component of  $G$  has even parity, then we also have  $\Gamma(G, b) \cong \Gamma(G) \times \mathbb{Z}_2$ .

**Lemma 2.8.** If the graph  $G$  has connected components  $G_1, \dots, G_k$ , then

$$\Gamma(G) = \Gamma(G_1) * \Gamma(G_2) * \dots * \Gamma(G_k).$$

Similarly to Lemma 2.6, Lemma 2.8 follows immediately from the presentation of  $\Gamma(G)$ .

**2.4. Consequences of main results for quantum strategies.** To explain the implications of Theorems 1.4 and 1.5 for perfect strategies of  $\mathcal{G}(G, b)$ , we summarize the following points from the proof of Theorem 2.4. Recall that a tracial state on a group  $G$  is a function  $\tau : G \rightarrow \mathbb{C}$  such that  $\tau(1) = 1$ ,  $\tau(ab) = \tau(ba)$  for all  $a, b \in G$ , and  $\tau$  is positive (meaning that  $\tau$  extends to a positive linear functional on the  $C^*$ -algebra of  $G$ ). The opposite group  $\Phi^{op}$  of a group  $(\Phi, \cdot)$  is the set  $\Phi$  with a new group operation  $\circ$  defined by  $a \circ b := b \cdot a$ .

**Proposition 2.9** ([CM14, CLS17]). Let  $(G, b)$  be a  $\mathbb{Z}_2$ -coloured graph.

- (a) Suppose  $\{\mathcal{X}_{ve}\}, \{\mathcal{Y}_{uf}\}, \nu$  is a perfect strategy for  $\mathcal{G}(G, b)$  on a Hilbert space  $\mathcal{H}$ . Let  $\mathcal{A}$  and  $\mathcal{B}$  be the subalgebras of  $B(\mathcal{H})$  generated by the observables  $\{\mathcal{X}_{ve}\}$  and  $\{\mathcal{Y}_{ve}\}$ , respectively. Then:

- If  $\mathcal{H}_0 := \overline{A\nu} \subset \mathcal{H}$ , then  $\mathcal{H}_0$  is also equal to  $\overline{B\nu}$ .
- If  $e \in E(G)$  has endpoints  $u, v$ , then the observable  $X_e := \mathcal{X}_{ve}|_{\mathcal{H}_0}$  on  $\mathcal{H}_0$  is also equal to  $\mathcal{X}_{ue}|_{\mathcal{H}_0}$ , and the mapping

$$x_e \mapsto X_e \text{ for } e \in E(G) \text{ and } J \mapsto -\mathbb{1}$$

defines a representation  $\phi$  of  $\mathcal{G}(G, b)$  on  $\mathcal{H}_0$ .

- There is also a unique representation  $\phi^R$  of the opposite group  $\Gamma(G, b)^{op}$  on  $\mathcal{H}_0$  defined by  $\phi^R(z)w\nu = w\phi(z)\nu$  for all  $z \in \Gamma(G, b)^{op}$  and  $w \in \mathcal{A}$ . Furthermore,  $\mathcal{Y}_{ve} = \phi^R(x_e)$  for all  $v \in V(G)$  and  $e \in E(G)$ .
- The function

$$\tau : \Gamma(G, b) \rightarrow \mathbb{C} : z \mapsto \langle \nu | \phi(z) \nu \rangle$$

defines a tracial state on  $\Gamma(G, b)$ .

- (b) Suppose  $\tau$  is a tracial state on  $\Gamma(G, b)$  with  $\tau(J) = -1$ . Then there is a Hilbert space  $\mathcal{H}$ , a unitary representation  $\phi$  of  $\Gamma(G, b)$  on  $\mathcal{H}$ , a unitary representation  $\phi^R$  of the opposite group  $\Gamma(G, b)^{op}$  on  $\mathcal{H}$ , and a quantum state  $\nu \in \mathcal{H}$  such that if  $\mathcal{X}_{ve} := \phi(x_e)$  and  $\mathcal{Y}_{ve} := \phi^R(x_e)$ , then  $\{\mathcal{X}_{ve}\}, \{\mathcal{Y}_{ve}\}$ ,  $\nu$  is a perfect commuting-operator strategy for  $\mathcal{G}(G, b)$ , and furthermore  $\tau(z) = \langle \nu | \phi(z) \nu \rangle$  for all  $z \in \Gamma(G, b)$  (so  $\tau$  is the tracial state of the strategy as in part (a) above).

- (c) Let  $\psi : \Gamma(G, b) \rightarrow U(\mathcal{H})$  be the left multiplication action of  $\Gamma(G, b)$  on  $\mathcal{H} = \ell^2\Gamma(G, b)$ , and let  $\nu = \frac{1-J}{\sqrt{2}} \in \mathcal{H}$ . If  $J \neq 1$  in  $\Gamma(G, b)$ , then the function

$$\tau : \Gamma(G, b) \rightarrow \mathbb{C} : z \mapsto \langle \nu | \psi(z) \nu \rangle$$

is a tracial state on  $\Gamma(G, b)$  with  $\tau(J) = -1$ .

- (d) Suppose  $\psi$  is a finite-dimensional representation of  $\Gamma(G, b)$  on  $\mathbb{C}^d$  with  $\psi(J) = -\mathbb{1}$ . Let  $\tau$  be the tracial state  $\tau(z) = \text{tr}(\phi(z))/d$  on  $\Gamma(G, b)$ . Then in part (b) we can take  $\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^d$ ,  $\phi = \psi \otimes \mathbb{1}$ ,  $\phi^R = \mathbb{1} \otimes \psi^T$  (where  $\psi^T$  refers to the transpose with respect to the standard basis on  $\mathbb{C}^d$ ), and  $\nu = \frac{1}{\sqrt{d}} \sum_{i=1}^d e_i \otimes e_i$ , where  $e_i$  is the  $i$ th standard basis element of  $\mathbb{C}^d$ .

*Proof.* Part (a) is [CLS17, Lemma 8]. Part (d) is proved in [CM14] for the more general class of binary constraint system games. Part (b) is slightly more general than what is shown in [CLS17], but if we take  $\phi$  and  $\phi^R$  to be the left and right GNS representations of  $\tau$ , and  $\nu$  to be the cyclic state for the GNS representation, then the rest of the proof is the same as for [CLS17, Theorem 4]. Part (c) is implicitly used in [CLS17], and follows immediately from the fact that  $J \neq 1$  (so that  $\nu$  is a unit vector) and  $J$  is central.  $\square$

Every group is isomorphic to its opposite group via the map  $\Phi \rightarrow \Phi^{op} : z \mapsto z^{-1}$ . Thus we can also think of the representation  $\phi^R$  of  $\Gamma(G, b)^{op}$  appearing in Proposition 2.9 as a representation of  $\Gamma(G, b)$ . We stick with  $\Gamma(G, b)^{op}$  to better distinguish the two representations.

Recall that perfect deterministic strategies correspond to solutions of  $\mathcal{I}(G)x = b$ , and hence to one-dimensional representations of  $\Gamma(G, b)$  with  $J = -1$ . Proposition 2.9 generalizes this by showing that there is a correspondence between perfect commuting-operator representations of  $\Gamma(G, b)$  and tracial states on  $\Gamma(G, b)$ . Thus, understanding the structure of  $\Gamma(G, b)$  allows us to understand the structure of perfect strategies for  $\mathcal{G}(G, b)$ . For instance, part (d) of Proposition 2.9 implies that every finite-dimensional irreducible representation  $\psi$  of  $\Gamma(G, b)$  with  $\psi(J) = -\mathbb{1}$

can be turned into a perfect quantum strategy for  $\mathcal{G}(G, b)$ . If  $\Gamma(G, b)$  is finite, then we can prove conversely that all perfect strategies are direct sums of strategies of this form:

**Corollary 2.10.** Suppose  $\Gamma(G, b)$  is a finite group, and let  $\phi_i : \Gamma(G, b) \rightarrow U(\mathcal{V}_i)$ ,  $i = 1, \dots, k$  be the irreducible unitary representations of  $\Gamma(G, b)$  with  $\phi_i(J) = -\mathbb{1}$ . Choose an orthonormal basis  $v_{i1}, \dots, v_{id_i}$  of  $\mathcal{V}_i$ , where  $d_i = \dim \mathcal{V}_i$ . Given  $w \in \Gamma(G, b)$ , let  $\phi_i(w)^T$  denote the transpose of  $\phi_i(w)$  with respect to the chosen basis, and let  $\phi_i^T$  denote the corresponding representation of  $\Gamma(G, b)^{op}$  on Hilbert space  $\mathcal{V}_i$ .

If  $\{\mathcal{X}_{ve}\}, \{\mathcal{Y}_{ve}\}$ ,  $\nu$  is a perfect commuting-operator strategy on a Hilbert space  $\mathcal{H}$ , then there is a finite-dimensional subspace  $\mathcal{H}_0 \subseteq \mathcal{H}$  which contains  $\nu$  and is invariant under  $\mathcal{X}_{ve}$  and  $\mathcal{Y}_{ve}$  for all  $v \in V(G)$ ,  $e \in E(G)$ , and an isometric isomorphism

$$I : \mathcal{H}_0 \rightarrow (\mathcal{V}_{i_1} \otimes \mathcal{V}_{i_1}) \oplus \dots \oplus (\mathcal{V}_{i_m} \otimes \mathcal{V}_{i_m}),$$

where  $1 \leq i_1 < \dots < i_m \leq k$ , such that

$$I(\nu) = \sum_{j=1}^m \frac{\lambda_j}{\sqrt{d_{i_j}}} \sum_{\ell=1}^{d_{i_j}} v_{i_j \ell} \otimes v_{i_j \ell}$$

for some positive real numbers  $\lambda_j$  with  $\sum_j \lambda_j^2 = 1$ , and

$$I\mathcal{X}_{ve}I^{-1} = \sum_{j=1}^m \phi_{i_j}(x_e) \otimes \mathbb{1}_{\mathcal{V}_{i_j}}, \quad I\mathcal{Y}_{ve}I^{-1} = \sum_{j=1}^m \mathbb{1}_{\mathcal{V}_{i_j}} \otimes \phi_{i_j}^T(x_e)$$

for all  $v \in V(G)$  and  $e \in E(v)$ .

*Proof.* Let  $\mathcal{H}_0$  be the subspace from part (a) of Proposition 2.9, and let  $\phi$  and  $\phi^R$  be the corresponding representations of  $\Gamma(G, b)$  and  $\Gamma(G, b)^{op}$  on  $\mathcal{H}_0$ . Since  $\Gamma(G, b)$  is finite,

$$\mathcal{H}_0 = \overline{\mathcal{A}\nu} = \overline{\phi(\mathbb{C}\Gamma(G, b))\nu} = \phi(\mathbb{C}\Gamma(G, b))\nu$$

is finite-dimensional. Since  $\phi(J) = -\mathbb{1}$ , there is an isometric isomorphism

$$I_0 : \mathcal{H}_0 \rightarrow \bigoplus_{i=1}^k \mathcal{V}_i \otimes \mathcal{W}_i$$

for some (possibly trivial) Hilbert spaces  $\mathcal{W}_1, \dots, \mathcal{W}_k$  with  $I_0\phi I_0^{-1} = \sum_{i=1}^k \phi_i \otimes \mathbb{1}$ . Let  $I_0\nu = \sum_{i=1}^k \nu_i$  where  $\nu_i \in \mathcal{V}_i \otimes \mathcal{W}_i$ . If  $P_i \in \mathbb{C}\Gamma(G, b)$  is the central projection for  $\mathcal{V}_i$ , then  $I_0\phi(P_i)I_0^{-1} = \phi_i(P_i) \otimes \mathbb{1} = \mathbb{1}_{\mathcal{V}_i} \otimes \mathbb{1}_{\mathcal{W}_i}$  is the projection onto  $\mathcal{V}_i \otimes \mathcal{W}_i$ , and so  $\nu_i = I_0\phi(P_i)\nu$ . Hence

$$\langle \nu_i | (\phi_i(z) \otimes \mathbb{1}) \nu_i \rangle = \langle \nu | \phi(z P_i) \nu \rangle$$

for all  $z \in \Gamma(G, b)$ , and since  $z \mapsto \langle \nu | \phi(z) \nu \rangle$  is tracial,

$$\tau_i : \Gamma(G, b) \rightarrow \mathbb{C} : z \mapsto \langle \nu_i | (\phi_i(z) \otimes \mathbb{1}) \nu_i \rangle$$

is a class function on  $\Gamma(G, b)$ , meaning that  $\tau_i(zw) = \tau_i(wz)$  for all  $z, w \in \Gamma(G, b)$ .

On the other hand, if we take the Schmidt decomposition

$$\nu_i = \sum_{j=1}^{m_i} c_{ij} u_{ij} \otimes w_{ij}$$

for some integer  $m_i \geq 0$ , positive real numbers  $c_{i1}, \dots, c_{im}$  and orthonormal subsets  $u_{i1}, \dots, u_{im_i}$  and  $w_{i1}, \dots, w_{im_i}$  of  $\mathcal{V}_i$  and  $\mathcal{W}_i$  respectively, then

$$\langle \nu_i | (\phi_i(z) \otimes \mathbb{1}) \nu_i \rangle = \sum_{j=1}^{m_i} c_{ij} \langle u_{ij} | \phi_i(z) u_{ij} \rangle.$$

Since  $\mathcal{V}_i$  is irreducible,  $\phi_i(\mathbb{C}\Gamma(G, b)) = \text{Lin}(\mathcal{V}_i)$ , the space of linear transformations from  $\mathcal{V}_i$  to itself. Thus the only way for  $\tau_i$  to be tracial is if  $\nu_i = 0$  (in which case  $m_i = 0$ ), or if  $m_i = \dim \mathcal{V}_i = d_i$  and  $c_{i1} = \dots = c_{id_i}$ . Since  $\phi(\mathbb{C}\Gamma(G, b))\nu = \mathcal{H}_0$ , we must also have

$$(\phi_i(\mathbb{C}\Gamma(G, b)) \otimes \mathbb{1}) \nu_i = I_0 \phi(\mathbb{C}\Gamma(G, b) P_i) \nu = I_0 \phi(P_i) \mathcal{H}_0 = \mathcal{V}_i \otimes \mathcal{W}_i.$$

and this is only possible if  $m_i = \dim \mathcal{W}_i$  as well.

Suppose that  $\nu_i \neq 0$ , and let

$$\gamma_i = \frac{1}{\sqrt{d_i}} \sum_{j=1}^{d_i} v_{ij} \otimes v_{ij} \in \mathcal{V}_i \otimes \mathcal{V}_i.$$

Recall that if  $A \in \text{Lin}(\mathcal{V}_i)$ , then  $(A \otimes \mathbb{1})\gamma_i = (\mathbb{1} \otimes A^T)\gamma_i$ , where the transpose is taken with respect to the basis  $v_{i1}, \dots, v_{id_i}$ . Let  $U_{i1} : \mathcal{V}_i \rightarrow \mathcal{V}_i$  be the unitary transformation sending  $v_{ij} \mapsto u_{ij}$ , and let  $U_{i2} : \mathcal{V}_i \rightarrow \mathcal{W}_i$  be the unitary transformation sending  $v_{ij} \mapsto w_{ij}$ . Then

$$\nu_i = \lambda_i (U_{i1} \otimes U_{i2}) \gamma_i = \lambda_i (\mathbb{1} \otimes (U_{i2} U_{i1}^T)) \gamma_i,$$

where  $\lambda_i := \sqrt{d_i} c_{i1}$ . Let  $1 \leq i_1 < \dots < i_m \leq k$  be the indices  $i$  such that  $\mathcal{W}_i \neq 0$ , and let

$$I_1 = \sum_{j=1}^m \mathbb{1} \otimes U_{i_j 2} U_{i_j 1}^T.$$

Since  $U_{i1}^T$  is unitary for all  $i$ ,  $I_1$  is an isometric isomorphism. Hence

$$I = I_1 I_0 : \mathcal{H}_0 \rightarrow \bigoplus_{j=1}^m \mathcal{V}_{i_j} \otimes \mathcal{V}_{i_j}$$

is an isometric isomorphism with

$$I\nu = \sum_{j=1}^m \lambda_{i_j} \gamma_{i_j} \text{ and } I\phi I^{-1} = \sum_{j=1}^m \phi_{i_j} \otimes \mathbb{1}.$$

Since  $\nu$  is a unit vector,  $\sum_{j=1}^m \lambda_{i_j}^2 = 1$ . Finally, by Proposition 2.9, part (a),

$$\begin{aligned} I\phi^R(z)I^{-1}(\phi_{i_j}(w) \otimes \mathbb{1})\lambda_{i_j}\gamma_{i_j} &= I\phi^R(z)\phi(wP_{i_j})\nu = I\phi(wP_{i_j})\phi(z)\nu \\ &= I\phi(wz)I^{-1}\lambda_{i_j}\gamma_{i_j} = (\phi_{i_j}(wz) \otimes \mathbb{1})\lambda_{i_j}\gamma_{i_j} \\ &= (\phi_{i_j}(w) \otimes \phi_{i_j}(z)^T)\lambda_{i_j}\gamma_{i_j} \end{aligned}$$

for all  $z \in \Gamma(G, b)^{op}$  and  $w \in \mathbb{C}\Gamma(G, b)$ . Thus we have

$$I\phi^R I^{-1} = \sum_{j=1}^m \mathbb{1} \otimes \phi_{i_j}^T$$

as required.  $\square$

**Remark 2.11.** In Corollary 2.10, we end up with a quantum strategy on a direct sum  $\bigoplus_i \mathcal{H}_i \otimes \mathcal{H}_i$  of tensor products of Hilbert spaces, with the first players' observables  $\mathcal{X}_{ve}$  acting on the first tensor factor, and the second players' observables  $\mathcal{Y}_{ve}$  acting on the second tensor factor. It is well-known that every quantum strategy (but not every commuting-operator strategy) for any nonlocal game can be put in this form, and this tensor product decomposition is often used explicitly in the definition of quantum strategies (see, e.g. [SW08]). To avoid stating two versions of all the results in this section, we've used a streamlined definition of quantum strategies that does not include an explicit tensor product decomposition. If we are given a strategy  $\{\mathcal{X}_{ve}\}, \{\mathcal{Y}_{ve}\}, \nu$  on a Hilbert space  $\mathcal{H}$  with an explicit tensor product decomposition, so  $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$  and  $\mathcal{X}_{ve} = \tilde{\mathcal{X}}_{ve} \otimes \mathbb{1}$ ,  $\mathcal{Y}_{ve} = \mathbb{1} \otimes \tilde{\mathcal{Y}}_{ve}$  for all  $v \in V(G)$ ,  $e \in E(G)$ , then it is possible to prove a slightly stronger version of Corollary 2.10 in which the subspace  $\mathcal{H}_0$  and the isometry  $I$  are local, meaning that  $\mathcal{H}_0 = \bigoplus_i \mathcal{H}_i^A \otimes \mathcal{H}_i^B$  for some subspaces  $\mathcal{H}_i^A$  and  $\mathcal{H}_i^B$  of  $\mathcal{H}^A$  and  $\mathcal{H}^B$  respectively, and  $I = \sum_i I_i^A \otimes I_i^B$  for isometries  $I_i^A$  and  $I_i^B$  acting on  $\mathcal{H}_i^A$  and  $\mathcal{H}_i^B$  respectively. Other variants—for instance, in which  $\mathcal{H}_0 = \mathcal{H}$  and  $I$  is an isometry but not necessarily an isomorphism—are also possible in this setting. For brevity, we leave these variants for the interested reader.

**Example 2.12.** Let  $b$  be an odd parity colouring of  $G$ , where  $G = K_{3,3}$  or  $K_5$ . Then  $\mathcal{G}(G, b)$  has no perfect deterministic strategy by Lemma 2.1. Since  $G$  is nonplanar, Theorem 1.5 implies that  $\Gamma(G, b)$  is nonabelian. In addition,  $G$  does not contain two disjoint cycles or  $K_{3,6}$  as a minor so by Theorem 1.4  $\Gamma(G, b)$  is finite. Using the mathematical software system SageMath [Sag17], we compute the character table for  $\Gamma(G, b)$  and find that both  $\Gamma(K_{3,3}, b)$  and  $\Gamma(K_5, b)$  have a unique four-dimensional irreducible unitary representation with  $J \mapsto -\mathbb{1}$ .

For each corresponding nonlocal game, a perfect quantum strategy can be obtained from the irreducible representation by the method outlined in part (d) of Proposition 2.9. Since  $\Gamma(G, b)$  is finite, the perfect quantum strategy for  $\mathcal{G}(G, b)$  is unique by Corollary 2.10, and hence these are robust self-tests by [CS17]. This is well-known: as mentioned in the introduction,  $\mathcal{G}(K_{3,3}, b)$  and  $\mathcal{G}(K_5, b)$  are the magic square and magic pentagram nonlocal games [WBMS16, KM18, CS17].

**Example 2.13.** Let  $b$  be an odd parity colouring of  $K_{3,4}$ , so that  $\mathcal{G}(K_{3,4}, b)$  does not have a perfect deterministic strategy by Lemma 2.1. Since  $K_{3,4}$  is nonplanar, Arkhipov's theorem implies that this game has a perfect quantum strategy.  $K_{3,4}$  does not contain  $K_{3,6}$  or two disjoint cycles as a minor, so  $\Gamma(K_{3,4}, b)$  is finite by Theorem 1.4. Since  $K_{3,4}$  contains  $K_{3,3}$  as a minor and  $b$  is odd, Theorem 1.5 implies that  $\Gamma(K_{3,4}, b)$  is nonabelian.

By direct computation, we find that  $\Gamma(K_{3,4}, b)$  has order 512. By computing the character table, we see that  $\Gamma(K_{3,4}, b)$  has 16 irreducible representations in which  $J \mapsto -\mathbb{1}$ , all of dimension 4. By part (d) of Proposition 2.9, there is a perfect quantum strategy corresponding to each of these irreducible representations. Corollary 2.10 shows that all perfect strategies are direct sums of these 16 strategies, with some weights  $\lambda_j$ . Notably, this gives an example of a self-testing result in which there is no single ideal strategy. Examples of self-tests with similar behaviour are given in [CMMN20, Kan20].

If  $b$  is an odd  $\mathbb{Z}_2$ -colouring of a connected nonplanar graph  $G$ , then Theorem 1.5 implies (and it is also not hard to see from Arkhipov's proof of Theorem 1.1)

that  $\Gamma(G, b)$  is nonabelian. Thus when  $b$  is odd,  $\Gamma(G, b)$  must be nonabelian for  $\mathcal{G}(G, b)$  to have a perfect quantum strategy (for any graph  $G$ ). On the other hand, when  $b$  is even,  $\mathcal{G}(G, b)$  has a perfect deterministic strategy. If  $\Gamma(G, b)$  is abelian, then  $\Gamma(G, b)$  is in fact finite, so as a special case of Corollary 2.10, every perfect strategy for  $\mathcal{G}(G, b)$  is a direct sum of deterministic strategies on the support of the state. As mentioned in the introduction, this property characterizes when  $\Gamma(G, b)$  is abelian:

**Corollary 2.14.** Let  $b$  be a  $\mathbb{Z}_2$ -colouring of a graph  $G$ , such that the restriction of  $b$  to each connected component of  $G$  has even parity. Let  $(\{f_v^{(i)}\}, \{g_v^{(i)}\})$ ,  $i = 1, \dots, k$  be a complete list of the deterministic strategies for  $\mathcal{G}(G, b)$ . Then  $\Gamma(G, b)$  is abelian if and only if for every perfect commuting-operator strategy  $\mathcal{H}$ ,  $\nu$ ,  $\{\mathcal{X}_{ve}\}$ ,  $\{\mathcal{Y}_{ve}\}$ , there is a subspace  $\mathcal{H}_0$  of  $\mathcal{H}$  which contains  $\nu$  and is invariant under  $\mathcal{X}_{ve}$  and  $\mathcal{Y}_{ve}$  for all  $v \in V(G)$ ,  $e \in E(G)$ , a sequence  $1 \leq i_1 < \dots < i_m \leq k$  and orthonormal basis  $\nu_1, \dots, \nu_m$  for  $\mathcal{H}_0$  such that  $\nu = \sum_{j=1}^m \lambda_j \nu_j$  for some positive real numbers  $\lambda_j$ , and

$$\mathcal{X}_{ve}|_{\mathcal{H}_0} = \sum_{j=1}^m (-1)^{f_v^{(i_j)}(e)} \nu_j \nu_j^*, \quad \mathcal{Y}_{ve}|_{\mathcal{H}_0} = \sum_{j=1}^m (-1)^{g_v^{(i_j)}(e)} \nu_j \nu_j^*,$$

for all  $v \in V(G)$ ,  $e \in E(G)$ .

*Proof.* When  $\Gamma(G, b)$  is abelian, the irreducible representations of  $\Gamma(G, b) = \Gamma(G, b)^{op}$  are exactly the one-dimensional representations. So the corollary follows immediately from Corollary 2.10 and the fact that one-dimensional representations of  $\Gamma(G, b)^{op} = \Gamma(G, b)$  with  $J = -1$  are the same as solutions of  $\mathcal{I}(G)x = b$ .

Conversely, if  $\mathcal{H}$ ,  $\nu$ ,  $\{\mathcal{X}_{ve}\}$ ,  $\{\mathcal{Y}_{ve}\}$  is a perfect commuting-operator strategy which is a direct sum of deterministic strategies as in the statement of the corollary, then the observables  $\mathcal{X}_{ve}|_{\mathcal{H}_0}$  and  $\mathcal{X}_{v'e'}|_{\mathcal{H}_0}$  commute for all  $v, v' \in V(G)$ ,  $e, e' \in E(G)$ . As a result, if  $\tau$  is the tracial state of this strategy from part (a) of Proposition 2.9, then  $\tau([x, y]) = 1$  for all  $x, y \in \Gamma(G, b)$ .

Suppose  $\Gamma(G, b)$  is nonabelian. Since the restriction of  $b$  to each connected component of  $G$  is even parity, Lemmas 2.5 and 2.6 imply that  $\Gamma(G, b) \cong \Gamma(G) \times \mathbb{Z}_2$ , where  $J = J_{G,b}$  is mapped to the generator of the  $\mathbb{Z}_2$  factor. Since  $\Gamma(G, b)$  is nonabelian,  $\Gamma(G)$  is nonabelian, and in this way we can find  $x, y \in \Gamma(G, b)$  such that  $[x, y] \notin \{1, J\}$ . Let  $\psi$  be the left action of  $\Gamma(G, b)$  on  $\mathcal{H} = \ell^2 \Gamma(G, b)$ , let  $\nu = \frac{1-J}{\sqrt{2}} \in \mathcal{H}$ , and let  $\tau(g) = \langle \nu | g \nu \rangle$  be the tracial state on  $\Gamma(G, b)$  from part (c) of Proposition 2.9. Using the definition of  $\tau$ , we see that  $\tau(g) = 0$  if  $g \notin \{1, J\}$ . In particular,  $\tau([x, y]) = 0$ . But by part (b) of Proposition 2.9, there is a perfect commuting-operator strategy with tracial state  $\tau$ . By the paragraph above, it is not possible for every perfect commuting-operator strategy of  $\mathcal{G}(G, b)$  to be a direct sum of deterministic strategies as in the statement of the corollary.  $\square$

**Example 2.15.** Let  $b$  be an even parity colouring of  $K_{3,3}$ . Since  $K_{3,3}$  does not contain  $C_2 \sqcup C_2$  or  $K_{3,4}$ , Theorem 1.5 implies that  $\Gamma(K_{3,3}, b)$  is abelian. By direct computation with SageMath, we find that  $\Gamma(K_{3,3}, b)$  has order 32, and hence has 32 distinct irreducible representations (all one-dimensional). Of these representations, 16 send  $J \mapsto -1$ , so  $\mathcal{G}(K_{3,3}, b)$  has 16 distinct deterministic strategies. By Corollary 2.14, every perfect strategy for  $\mathcal{G}(K_{3,3}, b)$  is a weighted direct sum of these 16 deterministic strategies.

**Example 2.16.** Let  $b$  be an even parity colouring of  $K_{3,4}$ . As in Example 2.13,  $\Gamma(K_{3,4}, b)$  is finite, and by Theorem 1.5,  $\Gamma(K_{3,4}, b)$  is not abelian. By Lemma 2.5,  $\Gamma(K_{3,4}, b) \cong \Gamma(K_{3,4}) \times \mathbb{Z}_2$ . By direct computation with SageMath, we see that  $\Gamma(K_{3,4})$  has order 256, 64 irreducible representations of dimension 1, and 12 irreducible representations of dimension 4. It follows that  $\Gamma(K_{3,4}, b)$  (which has order 512) has 64 irreducible representations of dimension 1 and 12 irreducible representations of dimension 4 sending  $J \mapsto -\mathbb{1}$ . By Corollary 2.10, every perfect strategy of  $\Gamma(K_{3,4}, b)$  is a direct sum of these irreducible representations. While the one-dimensional representations give deterministic perfect strategies of  $\mathcal{G}(K_{3,4}, b)$ , the four-dimensional representations yield perfect strategies which are not direct sums of deterministic strategies. Interestingly, the correlation matrices of these four-dimensional strategies are still classical.

### 3. GRAPH MINOR OPERATIONS FOR $\mathbb{Z}_2$ -COLOURED GRAPHS

Although we've motivated the definition of graph incidence groups via the connection with graph incidence games, graph incidence groups are fairly natural from the point of view of graph theory. For instance, the linear relations are a noncommutative generalization of the usual flow problem on the graph. The requirement that  $x_e$  and  $x_{e'}$  commute when  $e$  and  $e'$  are incident to a common vertex, which comes from the fact that  $x_e$  and  $x_{e'}$  correspond to jointly measurable observables, is also natural from the graph theory point of view, since there is no natural order on the vertices adjacent to a given vertex. In the rest of the paper, we focus on graph incidence groups from a combinatorial point of view, starting in this section with the proof of Lemma 1.2.

Before proving Lemma 1.2, we need to introduce a notion of minor operations for  $\mathbb{Z}_2$ -coloured graphs. Recall that the standard minor operations are

- (1) *edge deletion*, which takes a graph  $G$  and an edge  $e$ , and returns the graph  $G \setminus e$  with vertex set  $V(G)$  and edge set  $E(G) \setminus \{e\}$ ;
- (2) *edge contraction*, which takes a graph  $G$  and edge  $e$ , and returns the graph  $G/e$  with vertex set  $V(G) \setminus \{v_1, v_2\} \cup \{u\}$ , where  $v_1, v_2$  are the endpoints of  $e$  and  $u$  is a new vertex, and edge set  $E(G) \setminus (E(v_1) \cap E(v_2))$ , where any edge incident to  $v_1$  or  $v_2$  is now incident to  $u$ ; and
- (3) *vertex deletion*, which takes a graph  $G$  and vertex  $v$ , and returns the graph  $G \setminus v$  with vertex set  $V(G) \setminus \{v\}$  and edge set  $E(G) \setminus E(v)$ .

A graph  $H$  is said to be a graph minor of a graph  $G$  if  $H$  can be constructed from  $G$  by a sequence of graph minor operations. Notice that for edge contraction  $G/e$ , we remove all edges between the endpoints of  $e$  to avoid creating loops.

For  $\mathbb{Z}_2$ -coloured graphs, we start with the standard graph minor operations, but add an additional operation, and place a restriction on vertex deletion:

**Definition 3.1.** The graph minor operations for  $\mathbb{Z}_2$ -coloured graphs are

- (1) *edge deletion*, which takes a  $\mathbb{Z}_2$ -coloured graph  $(G, b)$  and an edge  $e$ , and returns the graph  $(G \setminus e, b)$ ;
- (2) *edge contraction*, which takes a  $\mathbb{Z}_2$ -coloured graph  $(G, b)$  and an edge  $e$ , and returns the graph  $(G/e, b')$  with

$$b'(v) = \begin{cases} b(v_1) + b(v_2) & v = u \\ b(v) & v \neq u \end{cases},$$

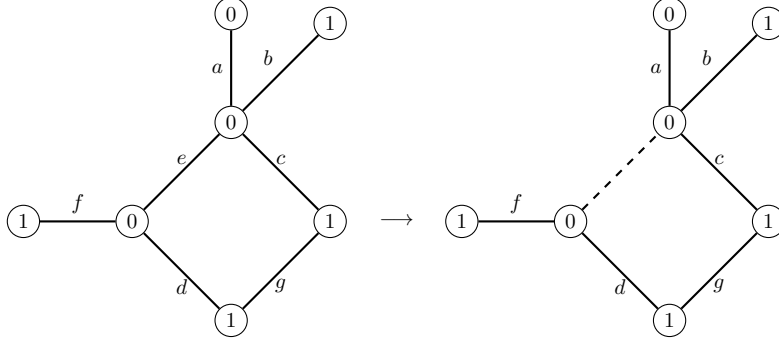


FIGURE 2. Deleting an edge  $e \in E(G)$  does not change the colour of vertices.

- where as above  $v_1$  and  $v_2$  are the endpoints of  $e$ , and  $u$  is the new vertex;
- (3) *vertex deletion*, which takes a  $\mathbb{Z}_2$ -coloured graph  $(G, b)$  and a vertex  $v$  with  $b(v) = 0$ , and returns the graph  $(G \setminus v, b|_{V(G) \setminus \{v\}})$ ; and
  - (4) *edge toggling*, which takes a  $\mathbb{Z}_2$ -coloured graph  $(G, b)$  and an edge  $e$ , and returns the graph  $(G, b')$  with

$$b'(v) = \begin{cases} b(v) & e \notin E(v) \\ b(v) + 1 & e \in E(v) \end{cases}.$$

A graph  $(H, c)$  is a *graph minor* of  $(G, b)$  if it can be constructed from  $(G, b)$  by a sequence of minor operations.

The minor operations for coloured graphs are shown in Figures 2, 3, 4, and 5. Note that all of the operations in Definition 3.1 preserve the parity of the colouring. In particular, vertices can only be deleted if they are labelled by 0, since otherwise deleting the vertex would change the parity. To delete a non-isolated vertex  $v$  coloured by 1, we can toggle an edge in  $E(v)$  to change the colour of  $v$  to 0. However, isolated vertices labelled by 1 cannot be deleted. Although this restriction is necessary for Lemma 1.2, it does cause a problem: if  $(G, b)$  is a  $\mathbb{Z}_2$ -coloured graph with two connected components  $(G_0, b_0)$  and  $(G_1, b_1)$ , and  $(H, c)$  is a minor of  $(G_0, b_0)$ , then  $(H, c)$  might not be a minor of  $(G, b)$  if  $b_1$  has odd parity. To work around this problem, we often restrict to connected graphs (see, for instance, Lemma 3.2).

For any fixed graph  $H$ , it is possible to decide whether  $H$  is a minor of  $G$  in time polynomial in the size of  $G$  [RS95]. Including edge toggling as a minor operation allows us to efficiently determine when a  $\mathbb{Z}_2$ -coloured graph  $(H, c)$  is a minor of  $(G, b)$ .

**Lemma 3.2.** Let  $(G, b)$  be a connected  $\mathbb{Z}_2$ -coloured graph. Then  $(H, c)$  is a minor of  $(G, b)$  if and only if  $H$  is a minor of  $G$  and the parity of  $c$  is equal to the parity of  $b$ .

*Proof.* Clearly if  $(H, c)$  is a minor of  $(G, b)$  then  $H$  is a minor of  $G$  and  $c$  has the same parity as  $b$ . For the converse, we first show that if  $b$  and  $b'$  are  $\mathbb{Z}_2$ -colourings of  $G$  with the same parity  $p$ , then it is possible to change  $b$  to  $b'$  by edge toggling. Indeed, let  $v$  be some vertex of  $G$ , and let  $b''$  be the  $\mathbb{Z}_2$ -colouring with  $b''(v) = p$



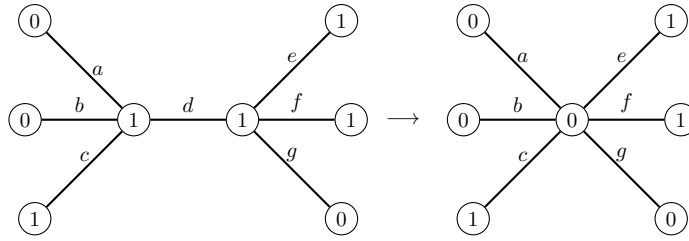


FIGURE 3. Contraction of an edge  $d \in E(G)$  with endpoints  $v_1$  and  $v_2$  results in a new vertex  $u$  with incident edges  $E(u) = \{E(v_1) \cup E(v_2)\} \setminus (E(v_1) \cap E(v_2))$ , and colour  $b(u) = b(v_1) + b(v_2)$ .

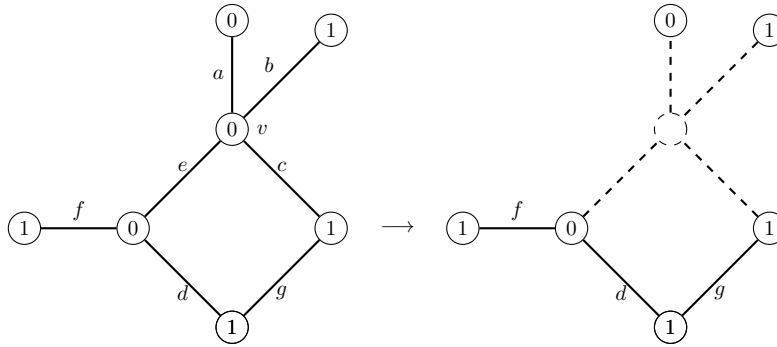


FIGURE 4. Deleting a vertex removes the vertex and all incident edges.

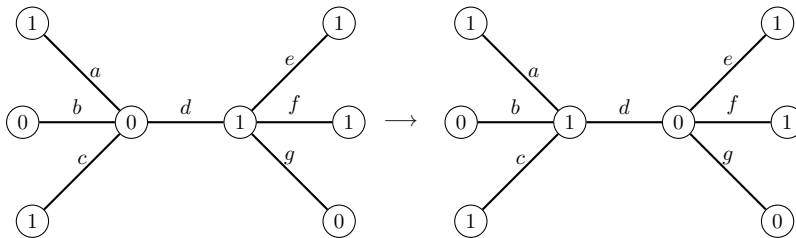


FIGURE 5. The edge toggle minor operation on the edge  $d$  toggles the  $\mathbb{Z}_2$ -colouring of the two endpoints.

and  $b''(w) = 0$  for  $w \neq v$ . Since  $G$  is connected, for every  $w \in V(G)$  there is a path  $P$  from  $v$  to  $w$ . Toggling all the edges in  $P$  adds the colour of  $w$  to the colour of  $v$ , changes the colour of  $w$  to 0, and leaves all other colours unchanged, so it is possible to change  $b$  to  $b''$  via edge toggling. Similarly, we can change  $b'$  to  $b''$ . Since edge toggling is reversible, we can also change  $b''$  to  $b'$  via edge toggling, and hence we can change  $b$  to  $b'$ .

Note that if  $H_1$  is the result of applying a minor operation to some graph  $H_0$ , then every vertex  $v$  of  $H_1$  comes from a vertex of  $H_0$ , in the sense that  $v$  is either a

vertex of  $H_0$  and is unaffected by the minor operation, or  $v$  is a new vertex added as a result of identifying two vertices of  $H_0$  during edge contraction.

Suppose that  $H$  is a minor of  $G$  and  $c$  has the same parity as  $b$ , and fix a sequence

$$G = G_0 \rightarrow G_1 \rightarrow \cdots \rightarrow G_k = H$$

of minor operations sending  $G$  to  $H$ . For each  $v \in V(H)$ , choose a vertex  $f(v) \in V(G)$  such that this sequence of minor operations eventually sends  $f(v)$  to  $v$ . Let  $b'$  be the colouring of  $G$  in which  $b'(w) = c(v)$  if  $w = f(v)$  for some  $v \in V(H)$ , and  $b'(w) = 0$  otherwise. Then  $b'$  has the same parity as  $c$  and  $b$ , and hence  $(G, b')$  is a minor of  $(G, b)$ . If the minor operation  $G_i \rightarrow G_{i+1}$  deletes the vertex  $w \in V(G_i)$ , and  $w' \in V(G)$  maps to  $w$  under the first  $i$  minor operations, then  $w'$  is not mapped to any vertex of  $H$ , so  $w' \neq f(v)$  for all  $v \in V(H)$ , and consequently  $b'(w') = 0$ . It follows that we can perform the  $\mathbb{Z}_2$ -coloured versions of each minor operation to turn  $(G, b')$  into  $(H, c)$ , and hence  $(H, c)$  is a minor of  $(G, b)$ .  $\square$

If  $(H, c)$  is a minor of a disconnected graph  $(G, b)$ , then it must be possible to decompose  $(H, c)$  into a disconnected union of (not necessarily connected) subgraphs  $(H_1, c_1), \dots, (H_k, c_k)$ , such that each subgraph  $(H_i, c_i)$  is a minor of a distinct connected component of  $(G, b)$ , and the remaining connected components of  $(G, b)$  have even parity. Thus we can determine whether  $(H, c)$  is a minor of  $(G, b)$  by going through all possible functions from connected components of  $(H, c)$  to connected components of  $(G, b)$ . This algorithm is polynomial in the size of  $G$ , although the exponent depends on the number of connected components of  $(H, c)$ .

We now turn to the following proposition, which describes how  $\Gamma(G, b)$  changes under minor operations.

**Proposition 3.3.** Let  $(G, b)$  be a  $\mathbb{Z}_2$ -coloured graph.

- (i) If  $e \in E(G)$  then there is a surjective homomorphism  $\phi : \Gamma(G, b) \rightarrow \Gamma(G \setminus e, b)$  sending

$$J \mapsto J \text{ and } x_f \mapsto \begin{cases} x_f & \text{if } f \neq e \\ 1 & \text{if } f = e \end{cases} \text{ for all } f \in E(G).$$

- (ii) If  $e \in E(G)$  is the only edge with endpoints  $v_1, v_2$ , then there is a surjective group homomorphism  $\phi : \Gamma(G, b) \rightarrow \Gamma(G/e, b')$  sending

$$J \mapsto J \text{ and } x_f \mapsto \begin{cases} x_f & \text{if } f \neq e \\ J^{b(v_1)} \prod_{e' \in E(v_1) \setminus \{e\}} x_{e'} & \text{if } f = e \end{cases} \text{ for all } f \in E(G),$$

where  $b'$  is the colouring of  $G/e$  described in part (2) of Definition 3.1.

- (iii) If  $v \in V(G)$  is an isolated vertex with  $b(v) = 0$ , then there is an isomorphism  $\phi : \Gamma(G, b) \rightarrow \Gamma(G \setminus v, b|_{V(G \setminus \{v\})})$  sending

$$J \mapsto J \text{ and } x_f \mapsto x_f \text{ for all } f \in E(G).$$

- (iv) If  $e \in E(G)$  then there is an isomorphism  $\phi : \Gamma(G, b) \rightarrow \Gamma(G, b')$  sending

$$J \mapsto J \text{ and } x_f \mapsto \begin{cases} x_f & \text{if } f \neq e \\ Jx_e & \text{if } f = e \end{cases} \text{ for all } f \in E(G),$$

where  $b'$  is the colouring described in part (4) of Definition 3.1.

*Proof.* In each case, we are given a homomorphism

$$\tilde{\phi} : \mathbb{F}(\{x_f : f \in E(G)\} \cup \{J\}) \rightarrow \mathbb{F}(\{x_f : f \in E(G')\} \cup \{J\})$$

for some  $\mathbb{Z}_2$ -coloured graph  $(G', b')$ , where  $\mathbb{F}(S)$  denotes the free group generated by  $S$ . For instance, in case (i),  $G' = G \setminus e$ ,  $b' = b$ , and  $\tilde{\phi}$  sends  $J \mapsto J$ ,  $x_f \mapsto x_f$  if  $f \neq e$ , and  $x_e \mapsto 1$ . In each case, we want to show that  $\tilde{\phi}$  descends to a homomorphism  $\phi : \Gamma(G, b) \rightarrow \Gamma(G', b')$ . To do this, we need to show that if  $r$  is a defining relation of  $\Gamma(G, b)$  from Definition 2.3, then  $\tilde{\phi}(r)$  is in the normal subgroup generated by the defining relations of  $\Gamma(G', b')$ . In case (i), it is clear that if  $r$  is a defining relation of  $\Gamma(G, b)$  not containing  $x_e$ , then  $\tilde{\phi}(r)$  is also a defining relation of  $\Gamma(G', b')$ . The relations containing  $x_e$  are  $x_e^2$ ,  $[x_e, x_f]$  for  $f \in E(v_1) \cup E(v_2)$ , and

$$r_i = J^{b(v)} \prod_{f \in E_G(v_i)} x_f, \quad i = 1, 2$$

where  $v_1, v_2$  are the endpoints of  $e$ . For the first two types of relations,  $\tilde{\phi}(x_e^2) = \tilde{\phi}([x_e, x_f]) = 1$ , while in the last case,  $\tilde{\phi}(r_1)$  and  $\tilde{\phi}(r_2)$  are again defining relations of  $\Gamma(G, b)$ . Hence  $\phi$  is well-defined. Since the image of  $\phi$  contains  $J$  and  $x_f$  for all  $f \in E(G')$ , in this case  $\phi$  is also surjective as required.

For (ii), once again if  $r$  is a defining relation of  $\Gamma(G, b)$  not containing  $x_e$ , then  $\tilde{\phi}(r)$  is a defining relation of  $\Gamma(G', b')$ . The generators  $x_f$ ,  $f \in E(v_1) \cup E(v_2) \setminus \{e\}$ , all commute in  $\Gamma(G', b')$ , so  $\tilde{\phi}([x_e, x_f]) = 1$  in  $\Gamma(G', b')$ . Also, since all these generators commute,

$$\tilde{\phi} \left( J^{b(v_1)} \prod_{f \in E(v_1)} x_f \right) = \left( J^{b(v_1)} \prod_{f \in E(v_1) \setminus \{e\}} x_f \right)^2 = 1$$

in  $\Gamma(G', b')$ , and  $\tilde{\phi}(x_e)^2 = 1$  in  $\Gamma(G', b')$  for the same reason. Finally, if  $u$  is the new vertex in  $G'$ , then

$$\begin{aligned} \tilde{\phi} \left( J^{b(v_2)} \prod_{f \in E(v_2)} x_f \right) &= \tilde{\phi}(x_e) \left( J^{b(v_2)} \prod_{f \in E(v_2) \setminus \{e\}} x_f \right) \\ &= J^{b(v_1)+b(v_2)} \prod_{f \in E(v_1) \cup E(v_2) \setminus \{e\}} x_f \\ &= J^{b(u)} \prod_{f \in E(u)} x_f = 1 \end{aligned}$$

in  $\Gamma(G', b')$ . So  $\phi$  is well-defined, and again the image of  $\phi$  contains  $J$  and  $x_f$  for all  $f \in E(G')$ , so  $\phi$  is surjective.

For (iii), an isolated vertex  $v$  with  $b(v) = 0$  corresponds to a trivial relation in the presentation of  $\Gamma(G, b)$ . Removing the vertex just deletes this relation, so  $\phi$  is clearly an isomorphism.

For (iv), toggling an edge does not change the parity of the  $\mathbb{Z}_2$ -colouring, and the map  $\phi$  is exactly the isomorphism between  $\Gamma(G, b)$  and  $\Gamma(G, b')$  defined in Lemma 2.5.  $\square$

The homomorphisms given in part (i) and (ii) of Proposition 3.3 are not necessarily isomorphisms. In part (i), the kernel of  $\phi : \Gamma(G, b) \rightarrow \Gamma(G \setminus e, b)$  is the normal

subgroup of  $\Gamma(G, b)$  generated by  $x_e$ . In part (ii), we have that

$$x_e = J^{b(v_1)} \prod_{e' \in E(v_1) \setminus \{e\}} x_{e'}$$

already in  $\Gamma(G, b)$ . However, if  $f \in E(v_1)$  and  $g \in E(v_2)$ , then  $x_f$  and  $x_g$  do not necessarily commute in  $\Gamma(G, b)$ , while  $\phi(x_f)$  and  $\phi(x_g)$  commute in  $\Gamma(G/e, b')$ . Thus the kernel of  $\phi : \Gamma(G, b) \rightarrow \Gamma(G/e, b')$  is generated by the commutators  $[x_f, x_g]$  for  $f \in E(v_1) \setminus \{e\}$ ,  $g \in E(v_2) \setminus \{e\}$ . Note as well that, although the homomorphism  $\phi$  in part (ii) seems to depend on a choice of endpoint  $v_1$  of  $e$ ,

$$J^{b(v_1)} \prod_{e' \in E(v_1) \setminus \{e\}} x_{e'} = J^{b(v_2)} \prod_{e' \in E(v_2) \setminus \{e\}} x_{e'}$$

in  $\Gamma(G/e, b')$ , so  $\phi$  is independent of the choice of endpoint.

The proof of Lemma 1.2 follows quickly from Proposition 3.3.

*Proof of Lemma 1.2.* Deleting a vertex  $v$  of  $(G, b)$  is equivalent to deleting all the edges in  $E(v)$ , and then deleting the now-isolated vertex  $v$ . Similarly, if  $e$  is an edge of  $(G, b)$  with endpoints  $v_1, v_2$ , then contracting  $e$  is equivalent to first deleting all the edges of  $(E(v_1) \cap E(v_2)) \setminus \{e\}$ , and then contracting  $e$ . Thus, if  $(H, c)$  is a minor of  $(G, b)$ , then there is a sequence of minor operations

$$(G, b) = (G_0, b_0) \rightarrow (G_1, b_1) \rightarrow \cdots \rightarrow (G_k, b_k) = (H, c)$$

sending  $(G, b)$  to  $(H, c)$ , where every operation is one of the operations listed in cases (i)-(iv) of Proposition 3.3. For these operations, there is a surjective homomorphism  $\Gamma(G_{i-1}, b_{i-1}) \rightarrow \Gamma(G_i, b_i)$  sending  $J_{G_{i-1}, b_{i-1}} \mapsto J_{G_i, b_i}$  for all  $1 \leq i \leq k$ . By composing these homomorphisms we get a surjective homomorphism  $\Gamma(G, b) \rightarrow \Gamma(H, c)$  sending  $J_{G, b} \mapsto J_{H, c}$ .  $\square$

As mentioned in the introduction, we can also prove a version of Lemma 1.2 for graph incidence groups of uncoloured graphs.

**Lemma 3.4.** If  $H$  is a minor of  $G$ , then there is a surjective morphism  $\Gamma(G) \rightarrow \Gamma(H)$ .

*Proof.* By Lemma 3.2,  $(H, 0)$  is a minor of  $(G, 0)$  (this is true even if  $G$  is disconnected, since every vertex is coloured 0), so there is a surjective morphism  $\Gamma(G, 0) \rightarrow \Gamma(H, 0)$  sending  $J_{G, 0} \mapsto J_{H, 0}$ , and since  $\Gamma(G) \cong \Gamma(G, 0) / \langle J_{G, 0} \rangle$  and  $\Gamma(H) \cong \Gamma(H, 0) / \langle J_{H, 0} \rangle$ , the lemma follows.  $\square$

Note that Lemmas 1.2 and 3.4 just require the existence of a homomorphism  $\Gamma(G, b) \rightarrow \Gamma(H, c)$  (resp.  $\Gamma(G) \rightarrow \Gamma(H)$ ), and this is all we will use in the remainder of the paper. However, Proposition 3.3 gives a recipe for finding this homomorphism from a sequence of minor operations. Let  $\mathbb{Z}_2$ -Graphs be the category of  $\mathbb{Z}_2$ -coloured graphs with morphisms freely generated by the minor operations of edge deletion, edge contraction of edges which are the only edges between their endpoints, vertex deletion of isolated vertices labelled by 0, and edge toggling. Then Proposition 3.3 implies that there is a functor  $\Gamma$  from  $\mathbb{Z}_2$ -Graphs to the category of groups over  $\mathbb{Z}_2$  with surjective homomorphisms. Similarly, if Graphs is the category of graphs with morphisms freely generated by edge deletion, edge contraction of edges which are the only edges between their endpoints, and vertex deletion of isolated vertices, then there is a functor  $\Gamma$  from Graphs to the category of groups with surjective homomorphisms. There is also a functor  $F$  from  $\mathbb{Z}_2$ -Graphs to Graphs

sending  $(G, b) \mapsto G$ , and another functor  $F'$  from groups over  $\mathbb{Z}_2$  with surjective homomorphisms to groups with surjective homomorphisms which sends  $(\Phi, J_\Phi)$  to  $\Phi/\langle J_\Phi \rangle$ , and these functors commute with  $\Gamma$ , i.e.  $\Gamma \circ F = F' \circ \Gamma$ .

**3.1. Forbidden minors for quotient closed properties.** A property  $P$  of graphs is minor-closed if  $G$  has  $P$  and  $H$  is a minor of  $G$ , then  $H$  also has  $P$ . Recall that the Robertson-Seymour theorem implies that if  $P$  is minor-closed, then there is a finite set of graphs  $\mathcal{F}$  which do not have  $P$ , and such that for any graph  $G$ ,  $G$  has  $P$  if and only if  $G$  does not contain any graph from  $\mathcal{F}$  as a minor. As an immediate corollary of this theorem and Lemma 3.4, we have:

**Corollary 3.5.** If  $P$  is a quotient closed property of groups, then there is a finite set  $\mathcal{F}$  of graphs such that for any graph  $G$ ,  $\Gamma(G)$  has  $P$  if and only if  $G$  avoids  $\mathcal{F}$ .

*Proof.* By Lemma 3.4, the graph property “ $\Gamma(G)$  has  $P$ ” is minor-closed.  $\square$

For technical reasons, it turns out to be difficult to extend Corollary 3.5 to all  $\mathbb{Z}_2$ -coloured graphs. A quasi-order  $\leq$  on a set  $X$  is said to be a well-quasi-order if for any infinite sequence  $x_1, x_2, \dots$  in  $X$ , there is a pair of indices  $1 \leq i < j$  such that  $x_i \leq x_j$ . If  $\leq$  is a well-quasi-order and  $x_1, x_2, \dots$  is an infinite sequence, then there must be a sequence  $1 \leq i_1 < i_2 < \dots$  of indices such that  $x_{i_1} \leq x_{i_2} \leq \dots$ . The full version of the Robertson-Seymour theorem states that the set of graphs is well-quasi-ordered under the minor containment relation (in which  $H \leq G$  if  $G$  contains  $H$  as a minor).

Say that  $(H, c) \leq (G, b)$  if  $(H, b)$  is a minor of  $(G, b)$ . The following example shows that  $\leq$  is not a well-quasi-order on the set of  $\mathbb{Z}_2$ -coloured graphs.

**Example 3.6.** Let  $G_n$  be the graph with  $n$  vertices and no edges, and let  $b_n : V(G_n) \rightarrow \{0, 1\}$  be the function sending every vertex to 1. Then no minor operation in Definition 3.1 can be applied to  $(G_n, b_n)$ , so  $(G_n, b_n)$  is not a minor of  $(G_k, b_k)$  for any  $n$  and  $k$ .

In the presentation of  $\Gamma(G, b)$ , an isolated vertex  $v$  with  $b(v) = 1$  corresponds to a relation  $J = 1$ . In the presentation of  $\Gamma(G_n, b_n)$ , this relation appears  $n$  times, and deleting all but one of these relations does not affect the group. This suggests adding an additional minor operation in which we can identify these connected components.

**Definition 3.7.** The minor operation *component identification* takes a  $\mathbb{Z}_2$ -coloured graph  $(G, b)$ , and two connected components  $G_i, i = 1, 2$ , of  $G$  which are isomorphic as  $\mathbb{Z}_2$ -coloured graphs, meaning that there is a graph isomorphism  $\phi : G_2 \rightarrow G_1$  such that  $b(\phi(v)) = b(v)$  for all  $v \in V(G_2)$ . It returns the graph with vertex set  $V(G) \setminus V(G_2)$ , edge set  $E(G) \setminus E(G_2)$ , and vertex colouring  $b|_{V(G) \setminus V(G_2)}$ .

We say that  $(H, c) \preceq (G, b)$  if  $(H, c)$  can be constructed from  $(G, b)$  using the minor operations in Definition 3.1, along with component identification.

It is not hard to see that if  $(H, c)$  is the result of identifying connected components  $G_1$  and  $G_2$  of  $(G, b)$ , and  $\phi : G_2 \rightarrow G_1$  is an isomorphism with  $b(\phi(v)) = b(v)$  for all  $v \in V(G_2)$ , then there is a surjective homomorphism

$$\Gamma(G, b) \mapsto \Gamma(H, c) : J \mapsto J \text{ and } x_e \mapsto \begin{cases} x_{\phi(e)} & e \in E(G_2) \\ x_e & e \notin E(G_2) \end{cases}.$$

Thus if  $(H, c) \preceq (G, b)$ , then there is a surjective homomorphism  $\Gamma(G, b) \rightarrow \Gamma(H, c)$ . Also, if  $(G_n, b_n)$  is the graph from Example 3.6, then  $(G_n, b_n) \preceq (G_k, b_k)$  for all  $n \leq k$ . Thus, it seems possible that  $\preceq$  is a well-quasi-order.

**Proposition 3.8.** The quasi-order  $\preceq$  is a well-quasi-order on the set of  $\mathbb{Z}_2$ -coloured graphs if and only if the set of minor-closed properties of connected graphs are well-quasi-ordered under inclusion.

*Proof.* Suppose  $(G, b)$  is a  $\mathbb{Z}_2$ -coloured graph, and let  $G_1, \dots, G_k$  be the connected components of  $G$ . Let  $b_i = b|_{V(G_i)}$ . For the purposes of this proof, we let  $G_{\text{even}}$  (resp.  $G_{\text{odd}}$ ) be the subgraph of  $G$  consisting of the connected components where  $b_i$  is even (resp.  $b_i$  is odd). Using Lemma 3.2, it is not hard to show that  $(H, c) \preceq (G, b)$  if and only if  $H_{\text{even}} \leq G_{\text{even}}$ ,  $H_{\text{odd}} \leq G_{\text{odd}}$ , and for every connected component  $G'$  of  $G_{\text{odd}}$ , there is a connected component  $H'$  of  $H_{\text{odd}}$  with  $H' \leq G'$ .

Suppose  $P$  is a minor-closed property of connected graphs. By the Robertson-Seymour theorem, there is a finite set  $\mathcal{F}$  of connected graphs such that  $G$  belongs to  $P$  if and only if  $H \not\leq G$  for all  $H \in \mathcal{F}$ . Conversely, if  $\mathcal{F}$  is a finite set of connected graphs, then “ $H \not\leq G$  for all  $H \in \mathcal{F}$ ” is a minor-closed property of connected graphs. Suppose  $P$  and  $P'$  are two minor-closed properties, defined by finite sets of connected graphs  $\mathcal{F}$  and  $\mathcal{F}'$  respectively. Then  $P$  is contained in  $P'$  if and only if

$$(3.1) \quad \text{for every } G \in \mathcal{F}', \text{ there is } H \in \mathcal{F} \text{ such that } H \leq G.$$

Indeed,  $P$  is contained in  $P'$  if and only if every graph not satisfying  $P'$  also does not satisfy  $P$ . So if  $P$  is contained in  $P'$ , and  $G \in \mathcal{F}'$ , then  $G$  does not satisfy  $P'$  and hence does not satisfy  $P$ . But this means that there is  $H \in \mathcal{F}$  with  $H \leq G$ . In the other direction, if  $\mathcal{F}$  and  $\mathcal{F}'$  satisfy Equation (3.1) and  $G$  does not satisfy  $P'$ , then there must be  $G' \in \mathcal{F}'$  such that  $G' \leq G$ , and hence there is  $H \in \mathcal{F}$  such that  $H \leq G' \leq G$ , so  $G$  does not satisfy  $P$ .

Suppose that  $P_1, P_2, \dots$  is a sequence of minor-closed properties of connected graphs, and let  $\mathcal{F}_1, \mathcal{F}_2, \dots$  be the corresponding sequence of forbidden minors. Let  $G_i$  be the graph with connected components  $\mathcal{F}_i$ , and let  $b_i$  be a  $\mathbb{Z}_2$ -colouring of  $G_i$  such that every connected component has odd parity. If  $\preceq$  is well-quasi-ordered, then there are indices  $1 \leq i < j$  such that  $(G_i, b_i) \preceq (G_j, b_j)$ . But then for every connected component  $G$  of  $(G_j)_{\text{odd}} = G_j$ , there must be a connected component  $H$  of  $(G_i)_{\text{odd}} = G_i$  with  $H \leq G$ . So  $\mathcal{F}_i$  and  $\mathcal{F}_j$  satisfy the condition in Equation (3.1), and  $P_i$  is contained in  $P_j$ . We conclude that minor-closed properties of connected graphs are well-quasi-ordered under inclusion.

On the other hand, suppose minor-closed properties of connected graphs are well-quasi-ordered under inclusion, and let  $(G_1, b_1), (G_2, b_2), \dots$  be a sequence of  $\mathbb{Z}_2$ -coloured graphs. Applying the fact that  $\leq$  is well-quasi-ordered to the sequence  $(G_1)_{\text{even}}, (G_2)_{\text{even}}, \dots$ , we see that there must be a sequence  $1 < i_1 < i_2 < \dots$  of indices such that  $(G_{i_1})_{\text{even}} \leq (G_{i_2})_{\text{even}} \leq \dots$ . Applying the same reasoning to the sequence  $(G_{i_1})_{\text{odd}}, (G_{i_2})_{\text{odd}}, \dots$ , we see that there must be a sequence of indices  $1 \leq j_1 < j_2 < \dots$  such that  $(G_{j_1})_{\text{even}} \leq (G_{j_2})_{\text{even}} \leq \dots$  and  $(G_{j_1})_{\text{odd}} \leq (G_{j_2})_{\text{odd}} \leq \dots$ . Let  $\mathcal{F}_k$  be the connected components of  $(G_{j_k})_{\text{odd}}$ , and let  $P_k$  be the corresponding minor-closed property of connected graphs. Then there is  $k < k'$  such that  $P_k$  is contained in  $P_{k'}$ , so that for every  $G \in \mathcal{F}_{k'}$ , there is  $H \in \mathcal{F}_k$  with  $H \leq G$ . But this means that  $(G_{j_k}, b_{j_k}) \preceq (G_{j_{k'}}, b_{j_{k'}})$ . We conclude that  $\preceq$  is a well-quasi-order.  $\square$

We are not aware of the inclusion order on minor-closed properties of connected graphs being studied in the literature. However, whether or not the inclusion order

on minor-closed properties of all graphs is well-quasi-ordered seems to be an open problem (see, for instance, [DK05, BNW10]), and we expect the same is true when looking at properties of connected graphs. Thus we do not know if  $\preceq$  is a well-quasi-order. There are also other graph operations which are natural with respect to the functor  $\Gamma(\cdot)$  (for instance, there is a generalization of component identification which takes  $G$  to  $H$  whenever  $G$  is a cover of  $H$ ). We leave it as an open problem to find a natural category of graph minor operations for  $\mathbb{Z}_2$ -coloured graphs, such that Lemma 1.2 holds, and such that graph minor containment is well-quasi-ordered.

Fortunately, the above technical problems disappear if we restrict to connected graphs.

**Corollary 3.9.** The graph minor containment relation  $\leq$  is a well-quasi-order on the set of connected  $\mathbb{Z}_2$ -coloured graphs.

*Proof.* Let  $(G_1, b_1), (G_2, b_2), \dots$  be an infinite sequence of  $\mathbb{Z}_2$ -coloured connected graphs. Then there must be an infinite sequence  $1 \leq i_1 < i_2 < \dots$  such that either  $b_{i_j}$  is odd for all  $j$ , or  $b_{i_j}$  is even for all  $j$ . Since  $\leq$  is a well-quasi-order on uncoloured graphs, there must be indices  $j < j'$  such that  $G_{i_j} \leq G_{i_{j'}}$ , and by Lemma 3.2,  $(G_{i_j}, b_{i_j}) \leq (G_{i_{j'}}, b_{i_{j'}})$ .  $\square$

Corollary 3.9 allows us to prove Corollary 1.3.

*Proof of Corollary 1.3.* By Lemma 1.2, “ $\Gamma(G, b)$  satisfies  $P$ ” is a minor-closed property  $P'$  of connected  $\mathbb{Z}_2$ -coloured graphs. Let  $\mathcal{S}$  be the set of connected  $\mathbb{Z}_2$ -coloured graphs not satisfying  $P'$ . By Corollary 3.9, there is a finite subset  $\mathcal{F}$  of  $\mathcal{S}$  such that every element of  $\mathcal{S}$  contains an element of  $\mathcal{F}$  as a minor. Conversely, since  $P'$  is minor-closed, any graph containing an element of  $\mathcal{F}$  as a minor cannot satisfy  $P'$ .  $\square$

**Remark 3.10.** The proof actually shows that there is a finite set  $\mathcal{F}$  of connected  $\mathbb{Z}_2$ -coloured graphs such that  $\Gamma(G, b)$  satisfies  $P$  if and only if  $(G, b)$  avoids  $\mathcal{F}$ . However, sometimes it is convenient to use disconnected graphs when writing down forbidden minors for connected graphs. For instance, in Theorems 1.4 and 1.5, it is conceptually simpler to use  $C_2 \sqcup C_2$  as a forbidden minor, although we could use a connected graph in its place.

#### 4. ARKHIPOV'S THEOREM AND PICTURES

Theorem 2.4 and Lemma 3.2 imply that Arkhipov's theorem (Theorem 1.1) can be restated in the following way:

**Theorem 4.1.** Let  $(G, b)$  be a connected  $\mathbb{Z}_2$ -coloured graph. Then the following are equivalent:

- (a)  $J_{G,b} = 1$  in  $\Gamma(G, b)$ .
- (b)  $J_{G,b}$  is trivial in finite-dimensional representations of  $\Gamma(G, b)$ .
- (c)  $(G, b)$  avoids  $(K_{3,3}, b')$  with  $b'$  odd,  $(K_5, b')$  with  $b'$  odd, and  $(K_1, b')$  with  $b'$  even.

Suppose that  $\phi : \Phi \rightarrow \Psi$  is a homomorphism of groups over  $\mathbb{Z}_2$ , so  $\phi(J_\Phi) = J_\Psi$ . If  $J_\Phi = 1$ , then  $J_\Psi = 1$ , so as mentioned in the introduction, “ $J_\Phi = 1$ ” is a quotient closed property of groups over  $\mathbb{Z}_2$ . Similarly, if  $\psi : \Psi \rightarrow U(\mathbb{C}^n)$  is a finite-dimensional representation of  $\Psi$  for some  $n \geq 1$  such that  $\psi(J_\Psi) \neq \mathbb{1}$ , then  $\psi \circ \phi$  is a finite-dimensional representation of  $\Phi$  with  $\psi \circ \phi(J_\Phi) \neq \mathbb{1}$ . So “ $J_\Phi$  is

trivial in finite-dimensional representations of  $\Phi$  is also a quotient closed property. As a result, Corollary 1.3 implies that both properties can be characterized by forbidden minors. However, Corollary 1.3 does not explain why these properties are equivalent, or why they are related to planarity of  $G$ . In this section, we show how to prove Theorem 4.1 in the group-theoretic language of Lemma 1.2, in a way that explains the equivalence of these two properties, and the relation to planarity.

For this proof, we recall the notion of *pictures* of groups. Pictures provide a graphical representation of relations in a group, and are a standard tool in combinatorial group theory [BRS07, Ol'12] (although the planar duals of pictures, called van Kampen diagrams, are more common). For solution groups of linear systems, pictures are particularly nice, since it is not necessary to keep track of the order of generators in the defining relations. A detailed definition of pictures for solution groups can be found in [Slo19b, Definition 7.2]. For the convenience of the reader, we give a streamlined version:

**Definition 4.2.** A *picture* over a graph  $G$  is an embedded planar graph  $\mathcal{P}$  with a distinguished vertex  $v_b$ , called the *boundary vertex*, in the outside face, as well as labelling functions  $h_E : E(\mathcal{P}) \rightarrow E(G)$  and  $h_V : V(\mathcal{P}) \setminus \{v_b\} \rightarrow V(G)$ , such that  $h_E|_{E(v)}$  is a bijection between  $E(v)$  and  $E(h_V(v))$  for all  $v \in V(\mathcal{P}) \setminus \{v_b\}$ .

A word  $e_1 \cdots e_k$  over  $E(G)$  is a *boundary word* of a picture  $\mathcal{P}$  if the edges incident to the boundary vertex are labelled by  $e_1, \dots, e_k$  when read in counterclockwise order from some starting edge. A picture is *closed* if  $E(v_b) = \emptyset$ . The *character* of a picture  $\mathcal{P}$  is the vector  $\chi(\mathcal{P}) \in \mathbb{Z}_2^{V(G)}$  with  $\chi(\mathcal{P})(v) = |h_V^{-1}(v)| \in \mathbb{Z}_2$ .

For simplicity, we use the same conventions for pictures as for graphs, in that multiedges are allowed, but loops are not. Note that if  $e_1 \cdots e_k$  is a boundary word of a picture  $\mathcal{P}$ , then every cyclic shift of this word is also a boundary word, since we can choose any starting edge.

When drawing pictures, we usually blow up the boundary vertex to a disk, and then think of the interior of this disk as the outside face. This gives a drawing of the picture inside a closed disk, with the boundary of the disk corresponding to the boundary vertex, as shown in Figures 6 and 7. Given such a drawing, we can shrink the boundary disk down to a vertex to get a drawing of the picture with the boundary vertex in the outside face, so these two ways of drawing a picture are equivalent.

Recall that a *graph homomorphism*  $\phi : G \rightarrow H$  is a function  $\phi_V : V(G) \rightarrow V(H)$  such that if  $v, w \in V(G)$  are adjacent in  $G$ , then  $\phi_V(v)$  and  $\phi_V(w)$  are adjacent in  $H$ . In particular,  $\phi_V(v) \neq \phi_V(w)$  if  $v$  and  $w$  are adjacent in  $G$ . If  $G$  and  $H$  do not have multiple edges between vertices, then given a graph homomorphism  $\phi_V : G \rightarrow H$ , we can define a function  $\phi_E : E(G) \rightarrow E(H)$  by sending  $e \in E(G)$  with endpoints  $v, w \in V(G)$  to  $e' \in E(H)$  with endpoints  $\phi_V(v)$  and  $\phi_V(w)$ . A graph homomorphism between graphs without multiple edges is a *cover* if  $\phi_E|_{E(v)}$  is a bijection from  $E(v)$  to  $E(\phi_V(v))$  for all  $v \in V(G)$ . To extend this concept to graphs with multiple edges between vertices, we say that a *cover* of a graph  $H$  is a homomorphism  $\phi_V : G \rightarrow H$  along with a function  $\phi_E : E(G) \rightarrow E(H)$ , such that if  $e \in E(G)$  has endpoints  $v, w$  then  $\phi_E(e)$  has endpoints  $\phi_V(v), \phi_V(w)$ , and such that  $\phi_E|_{E(v)}$  is a bijection from  $E(v)$  to  $E(\phi_V(v))$  for all  $v \in V(G)$ . A *planar cover* of  $H$  is a cover  $G \rightarrow H$  in which  $G$  is planar. An *embedded planar cover* of  $H$  is a planar cover  $G \rightarrow H$  along with a choice of planar embedding of  $G$ . By thinking of  $G$  as embedded in a closed disk (or equivalently, by adding a boundary vertex



to the outer face), any embedded planar cover  $G \rightarrow H$  can be regarded as a closed picture over  $H$  with labelling functions  $\phi_V$  and  $\phi_E$ .

Conversely, if  $\mathcal{P}$  is a closed picture over  $H$ , then all the data of  $\mathcal{P}$  is contained in the embedded planar graph  $\mathcal{P} \setminus v_b$ . If  $e \in E(\mathcal{P})$  has endpoints  $v, w \in V(\mathcal{P})$ , then  $h_E(e)$  must be incident to  $h_V(v)$  and  $h_V(w)$ . However, this does not necessarily imply that  $h_V(v)$  and  $h_V(w)$  are adjacent, since  $h_V(v)$  and  $h_V(w)$  could be equal. As a result,  $h_V$  might not be a homomorphism, with the consequence that  $\mathcal{P} \setminus v_b$  is not necessarily an embedded planar cover of  $H$ . Thus we can think of pictures as generalizations of planar covers, which preserve incidence rather than adjacency. A closed picture  $\mathcal{P}$  comes from an embedded planar cover if and only if  $h_V$  is a homomorphism, which happens if and only if  $h_V(v) \neq h_V(w)$  whenever vertices  $v$  and  $w$  are adjacent in  $\mathcal{P}$ .

The following lemma shows that boundary words of pictures give relations in the graph incidence group, and that all relations in the group arise in this way. This lemma is essentially due to van Kampen [VK33]. A proof of this version of the lemma can be found in [Slo19b].

**Lemma 4.3** (van Kampen lemma). Let  $(G, b)$  be a  $\mathbb{Z}_2$ -coloured graph. Then  $x_{e_1} \cdots x_{e_k} = J^a$  in  $\Gamma(G, b)$  if and only if  $e_1 \cdots e_k$  is the boundary word of a picture  $\mathcal{P}$  over  $G$  with  $\chi(\mathcal{P}) \cdot b = a$ . In particular,  $J = 1$  if and only if there is a closed picture  $\mathcal{P}$  with  $\chi(\mathcal{P}) \cdot b = 1$ .

In this lemma, if  $b, b' \in \mathbb{Z}_2^{V(G)}$ , then  $b' \cdot b := \sum_{v \in V(G)} b'(v)b(v)$ .

As an example of Lemma 4.3, we prove the following lemma:

**Lemma 4.4.** Let  $G$  be  $K_{3,3}$  or  $K_5$ , and let  $b$  be a  $\mathbb{Z}_2$ -colouring of  $G$  of parity  $a$ . If  $e$  and  $f$  are two edges of  $G$  which are not incident to a common vertex, then

$$[x_e, x_f] = J^a$$

in  $\Gamma(G, b)$ .

*Proof.* It is not hard to see that  $G$  has a planar drawing with a single crossing between edges  $e$  and  $f$ . If we replace the crossing point with a new boundary vertex  $v_b$ , we get a  $G$ -picture  $\mathcal{P}$  with boundary word  $efef$ , as shown in Figures 6 and 7.

Because every vertex of  $G$  occurs exactly once in the picture, the character  $\chi(\mathcal{P})$  is the vector of all 1's in  $\mathbb{Z}_2^{V(G)}$ . Thus  $\chi(\mathcal{P}) \cdot b = \sum_{v \in V(G)} b(v) = a$ , and the conclusion follows from Lemma 4.3.  $\square$

To prove Theorem 4.1, we need to know that if  $G = K_{3,3}$  or  $K_5$ , and  $b$  is an odd parity colouring, then  $\Gamma(G, b)$  is finite and  $J \neq 1$ . This can be done directly on a computer, as in Example 2.12. There is also a nice expression for these groups due to [CS17], which we now explain. Recall that the dihedral group  $\text{Dih}_n$  is the group with presentation

$$(4.1) \quad \text{Dih}_n = \langle z_1, z_2 : z_1^2 = z_2^2 = (z_1 z_2)^n = 1 \rangle.$$

$\text{Dih}_n$  is a finite group of order  $2n$ , and is nonabelian for  $n \geq 3$ . If  $n$  is even, then the center of  $\text{Dih}_n$  has a single non-trivial element  $(z_1 z_2)^{n/2}$ . When  $n = 4$ , this central element has order 2.

Let  $\psi : Z(\Psi_1) \rightarrow Z(\Psi_2)$  be an isomorphism between the centers of two groups  $\Psi_1$  and  $\Psi_2$ . The central product of  $\Psi_1$  and  $\Psi_2$  is the quotient of the product  $\Psi_1 \times \Psi_2$

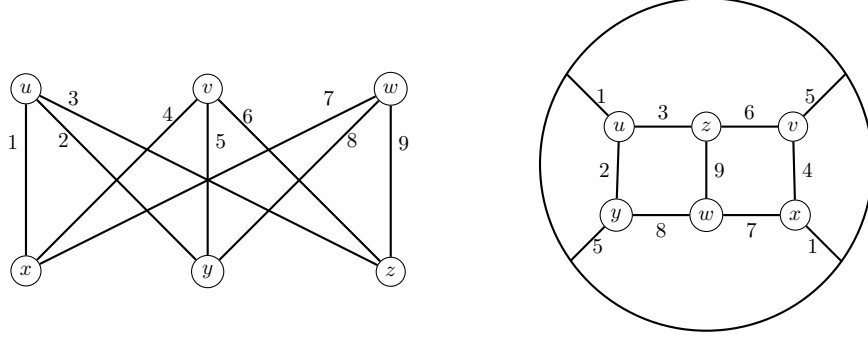


FIGURE 6. On the left is  $K_{3,3}$  with edges labelled from 1 to 9, and vertices labelled from  $u$  to  $z$ . On the right is a  $K_{3,3}$ -picture with boundary word 1 5 1 5.

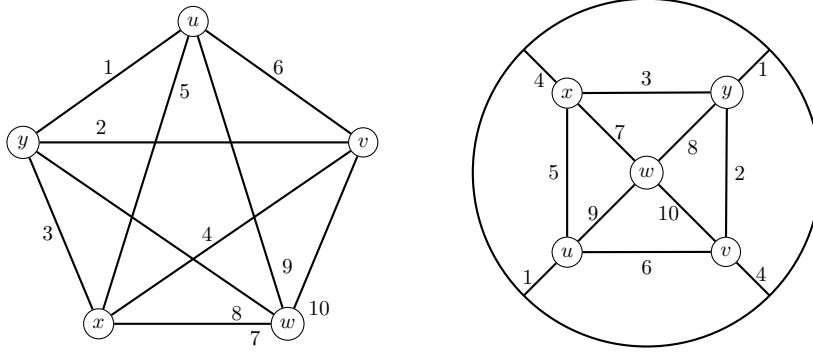


FIGURE 7. On the left is  $K_5$  with edges labelled from 1 to 10, and vertices labelled from  $u$  to  $y$ . On the right is a  $K_5$ -picture with boundary word 1 4 1 4.

by the normal subgroup generated by  $(z, \psi^{-1}(z))$  for  $z \in Z(\Psi_1)$ . We denote the central product by  $\Psi_1 \odot_{\psi} \Psi_2$ , or  $\Psi_1 \odot \Psi_2$  if the isomorphism  $\psi$  is clear. The groups  $\Psi_1$  and  $\Psi_2$  are naturally subgroups of  $\Psi_1 \odot \Psi_2$ , and the center of  $\Psi_1 \odot \Psi_2$  is  $Z(\Psi_1) = Z(\Psi_2)$ , considered as a subgroup of  $\Psi_1 \odot \Psi_2$ .

**Proposition 4.5** ([CS17]).

- (a) Let  $b$  be an odd parity colouring of  $K_{3,3}$ . Then

$$\Gamma(K_{3,3}, b) \cong \text{Dih}_4 \odot \text{Dih}_4$$

via an isomorphism which sends  $J \in \Gamma(K_{3,3}, b)$  to the unique non-trivial central element of  $\text{Dih}_n \odot \text{Dih}_n$ .

- (b) Let  $b$  be an odd parity colouring of  $K_5$ . Then

$$\Gamma(K_5, b) \cong \text{Dih}_4 \odot \text{Dih}_4 \odot \text{Dih}_4$$

via an isomorphism which sends  $J \in \Gamma(K_5, b)$  to the unique non-trivial central element of  $\text{Dih}_4 \odot \text{Dih}_4 \odot \text{Dih}_4$ .

Although a proof of Proposition 4.5 can be found in [CS17], we provide a proof of this proposition for the convenience of the reader.

*Proof.* For part (a), we use the vertex and edge labelling of  $K_{3,3}$  shown in Figure 6. Let  $b$  be the colouring with  $b(w) = 1$  and  $b(t) = 0$  for all other vertices  $t \neq w$ . Since the edges of  $K_{3,3}$  are labelled from 1 to 9, the group  $\Gamma(K_{3,3}, b)$  is generated by  $x_1, \dots, x_9$ . Using the presentation of  $\text{Dih}_4$  from Equation (4.1), we see that

$$\begin{aligned} \text{Dih}_4 \odot \text{Dih}_4 = \langle & z_{11}, z_{12}, z_{21}, z_{22} : z_{ij}^2 = 1 \text{ for all } i, j \in \{1, 2\}, \\ & (z_{i1}z_{i2})^4 = 1 \text{ for } i = 1, 2, \\ & [z_{1i}, z_{2j}] = 1 \text{ for all } i, j \in \{1, 2\}, \\ & (z_{11}z_{12})^2 = (z_{21}z_{22})^2 \rangle. \end{aligned}$$

Now we can define a homomorphism

$$\phi : \text{Dih}_4 \odot \text{Dih}_4 \rightarrow \Gamma(K_{3,3}, b)$$

by setting  $\phi(z_{11}) = x_1$ ,  $\phi(z_{12}) = x_5$ ,  $\phi(z_{21}) = x_2$ , and  $\phi(z_{22}) = x_4$ . To see that  $\phi$  is a homomorphism, observe that  $x_i^2 = 1$  for all  $1 \leq i \leq 9$ . Also, edges 2 and 4 both share common vertices with edges 1 and 5, so  $x_2$  and  $x_4$  both commute with  $x_1$  and  $x_5$ . Since 1 and 5 are not incident with a common vertex,  $(x_1x_5)^2 = J$  by Lemma 4.4. As a result,  $(x_1x_5)^4 = 1$ . Similarly,  $(x_2x_4)^2 = J$ , so  $(x_2x_4)^4 = 1$ , and  $(x_1x_5)^2 = (x_2x_4)^2$ . Thus  $x_1, x_5, x_2$ , and  $x_4$  satisfy the defining relations for  $\text{Dih}_4 \odot \text{Dih}_4$ , and hence  $\phi$  is well-defined.

To see that  $\phi$  is an isomorphism, we define

$$\phi^{-1} : \text{Dih}_4 \odot \text{Dih}_4 \rightarrow \Gamma(K_{3,3}, b)$$

by setting

$$(4.2) \quad \begin{aligned} \phi^{-1}(x_1) &= z_{11}, & \phi^{-1}(x_2) &= z_{21}, & \phi^{-1}(x_3) &= z_{11}z_{21}, \\ \phi^{-1}(x_4) &= z_{22}, & \phi^{-1}(x_5) &= z_{12}, & \phi^{-1}(x_6) &= z_{12}z_{22}, \\ \phi^{-1}(x_7) &= z_{11}z_{22}, & \phi^{-1}(x_8) &= z_{12}z_{21}, & \phi^{-1}(x_9) &= z_{11}z_{12}z_{21}z_{22}, \text{ and} \\ \phi^{-1}(J) &= (z_{11}z_{12})^2 = (z_{21}z_{22})^2. \end{aligned}$$

We can check that  $\phi^{-1}(r) = 1$  for all defining relations  $r$  of  $\Gamma(K_{3,3}, b)$ , so  $\phi^{-1}$  is well-defined as a homomorphism. For instance,

$$\phi^{-1}(x_7x_8x_9) = (z_{11}z_{12}z_{11}z_{12})(z_{22}z_{21}z_{21}z_{22}) = (z_{11}z_{12})^2 = \phi^{-1}(J),$$

while

$$\phi^{-1}(x_3x_6x_9) = (z_{11}z_{12}z_{11}z_{12})(z_{21}z_{22}z_{21}z_{22}) = (z_{11}z_{12})^2(z_{21}z_{22})^2 = 1,$$

matching the colouring  $b(w) = 1$  and  $b(z) = 0$ . Thus  $\phi$  is an isomorphism. By Lemma 2.5, part (a) is true for any other odd parity colouring of  $K_{3,3}$ .

The proof of part (b) is similar. We use the vertex and edge labelling of  $K_5$  from Figure 7, and let  $b$  be the  $\mathbb{Z}_2$ -colouring with  $b(w) = 1$  and  $b(t) = 0$  for vertices  $t \neq w$ . The group  $G = \text{Dih}_4 \odot \text{Dih}_4 \odot \text{Dih}_4$  has presentation

$$\begin{aligned} G = \langle & z_{ij}, i \in \{1, 2, 3\}, j \in \{1, 2\} : z_{ij}^2 = 1 \text{ for all } i \in \{1, 2, 3\}, j \in \{1, 2\}, \\ & (z_{i1}z_{i2})^4 = 1 \text{ for all } i \in \{1, 2, 3\}, \\ & [z_{ij}, z_{kl}] = 1 \text{ for all } i \neq k \in \{1, 2, 3\}, j, l \in \{1, 2\}, \\ & (z_{11}z_{12})^2 = (z_{21}z_{22})^2 = (z_{31}z_{32})^2 \rangle. \end{aligned}$$

To define an isomorphism  $\psi : G \rightarrow \Gamma(K_5, b)$ , we set

$$\psi(z_{11}) = x_1, \psi(z_{12}) = x_4, \psi(z_{21}) = x_2, \psi(z_{22}) = x_5, \psi(z_{31}) = x_3, \psi(z_{32}) = x_6.$$

That  $\phi$  is well-defined follows from the same arguments as in part (a); in particular, we once again use Lemma 4.4 to see that  $(x_1x_4)^2 = (x_2x_5)^2 = (x_3x_6)^2 = J$ . For the inverse, we define  $\phi^{-1} : \Gamma(K_5, b) \rightarrow G$  as the inverse of  $\phi$  on  $x_1, \dots, x_6$ , and setting

$$\begin{aligned} \phi^{-1}(x_7) &= \phi^{-1}(x_3x_4x_5) = z_{12}z_{22}z_{31}, & \phi^{-1}(x_8) &= \phi^{-1}(x_1x_2x_3) = z_{11}z_{21}z_{31}, \\ \phi^{-1}(x_9) &= \phi^{-1}(x_1x_5x_6) = z_{11}z_{22}z_{32}, & \phi^{-1}(x_{10}) &= \phi^{-1}(x_2x_4x_6) = z_{12}z_{21}z_{32}, \text{ and} \\ \phi^{-1}(J) &= (z_{11}z_{12})^2 = (z_{21}z_{22})^2 = (z_{31}z_{32})^2. \end{aligned}$$

The defining relations of  $\Gamma(K_5, b)$  for vertices  $u, v, x$ , and  $y$  follow immediately from the definition, while for vertex  $w$  we have

$$\phi^{-1}(x_7x_8x_9x_{10}) = (z_{12}z_{11}z_{11}z_{12})(z_{22}z_{21}z_{22}z_{21})(z_{31}z_{31}z_{32}z_{32}) = (z_{21}z_{22})^2 = \phi^{-1}(J),$$

which matches the colouring  $b(w) = 1$ . So  $\phi^{-1}$  is well-defined as a homomorphism, and hence  $\phi$  is an isomorphism.  $\square$

Proposition 4.5 also allows us to determine  $\Gamma(K_{3,3})$  and  $\Gamma(K_5)$ . This will be used in the next section.

**Corollary 4.6.**

- (a)  $\Gamma(K_{3,3}) = \mathbb{Z}_2^4$ , and if  $b$  is an even parity colouring then  $\Gamma(K_{3,3}, b) = \mathbb{Z}_2^5$ .
- (b)  $\Gamma(K_5) = \mathbb{Z}_2^6$ , and if  $b$  is an even parity colouring then  $\Gamma(K_5, b) = \mathbb{Z}_2^7$ .

*Proof.*  $\Gamma(K_{3,3}) = \Gamma(K_{3,3}, b)/\langle J \rangle$  for any colouring  $b$  of  $K_{3,3}$ . Take  $b$  to be an odd parity colouring, so that  $\Gamma(K_{3,3}, b) = \text{Dih}_4 \odot \text{Dih}_4$  has the presentation from Equation (4.2). Setting  $J = (z_{11}z_{12})^2 = 1$  in this presentation, we get that

$$\begin{aligned} \Gamma(K_{3,3}) &= \langle z_{ij}, i, j \in \{1, 2\} : (z_{ij})^2 = 1 \text{ for all } i, j \in \{1, 2\} \\ &\quad [z_{ij}, z_{kl}] = 1 \text{ for all } i, j, k, l \in \{1, 2\} \rangle = \mathbb{Z}_2^4. \end{aligned}$$

If  $b$  is even parity, then by Lemma 2.5,  $\Gamma(K_{3,3}, b) = \Gamma(K_{3,3}) \times \mathbb{Z}_2 = \mathbb{Z}_2^5$ . The proof of (b) is similar.  $\square$

*Proof of Theorem 4.1.* Let  $(G, b)$  be a  $\mathbb{Z}_2$ -coloured connected graph. Suppose that  $(G, b)$  satisfies (c), so that  $(G, b)$  avoids  $(K_5, b')$  with  $b$  odd,  $(K_{3,3}, b')$  with  $b$  odd, and  $(K_1, b')$  with  $b'$  even.  $G$  contains  $K_1$  as a minor, so by Lemma 2.5,  $b$  must be odd, and  $G$  cannot contain  $K_5$  or  $K_{3,3}$  as a minor. But this implies that  $G$  is planar. Choosing an embedding of  $G$  in a closed disk, and setting  $h_V(v) = v$  and  $h_E(e) = e$ , we get a closed picture  $\mathcal{P}$  of  $G$  with character  $\chi(\mathcal{P})$  equal to the vector of all 1's. Hence  $\chi(\mathcal{P}) \cdot b = 1$ , the parity of  $b$ , and by Lemma 4.3, we have  $J = 1$  in  $\Gamma(G, b)$ . Then  $J$  is also trivial in finite-dimensional representations of  $\Gamma(G, b)$ , so  $(G, b)$  satisfies (a) and (b).

Suppose that  $(G, b)$  does not satisfy (c). If  $(G, b)$  contains  $(K_1, b')$  with  $b'$  even, then by Lemma 3.2 the colouring  $b$  must also have even parity. By Lemma 2.5, there is an isomorphism  $\Gamma(G, b) \cong \Gamma(G, 0) \cong \Gamma(G) \times \mathbb{Z}_2$  sending  $J_{G,b}$  to the generator of the  $\mathbb{Z}_2$  factor. Composing with the projection  $\Gamma(G) \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$  and identifying  $\mathbb{Z}_2$  with the subgroup  $\pm 1 \subset U(\mathbb{C}^1)$ , we see that  $J$  is non-trivial in finite-dimensional representations of  $\Gamma(G, b)$ .

If  $(G, b)$  contains  $(H, b')$  where  $H = K_5$  or  $K_{3,3}$  and  $b'$  has odd parity, then by Lemma 1.2 there is a homomorphism  $\phi : \Gamma(G, b) \rightarrow \Gamma(H, b')$  with  $\phi(J_{G,b}) = J_{H,b'}$ . By Proposition 4.5,  $\Gamma(H, b')$  is a finite group and  $J_{H,b'} \neq 1$ . Since  $\Gamma(H, b')$  is finite, it has a faithful finite-dimensional representation, and composing with this representation, we see that  $J_{G,b}$  is non-trivial in finite-dimensional representations of  $\Gamma(G, b)$ .

In both cases, if  $(G, b)$  does not satisfy (c), then  $(G, b)$  does not satisfy (b). Since (a) implies (b), conditions (a), (b), and (c) are equivalent.  $\square$

Another equivalent formulation of Theorem 4.1 is that a graph  $G$  is planar if and only if there is a picture  $\mathcal{P}$  with character  $\chi(\mathcal{P}) \neq 0$ . Indeed, if there is a picture  $\mathcal{P}$  over  $G$  with  $\chi(\mathcal{P})(v) = 1$ , then let  $b$  be the  $\mathbb{Z}_2$ -colouring of  $G$  with  $b(v) = 1$ , and  $b(t) = 0$  for all vertices  $t \neq v$ . Then  $\chi(G) \cdot b = 1$ , so  $J = 1$  in  $\Gamma(G, b)$ , and  $G$  must be planar by Theorem 4.1. Conversely, if  $G$  is planar, then  $G$  itself can be turned into a closed picture  $\mathcal{P}$  with  $\chi(\mathcal{P})$  equal to the vector of all 1's.

More generally, any planar cover  $\mathcal{P}$  of  $G$  can be turned into a closed picture over  $G$  as discussed above. If  $G$  is connected and  $\phi : \mathcal{P} \rightarrow G$  is a planar cover of  $G$ , then the function  $|\phi_V^{-1}(v)|$  is a constant function of  $v \in V(G)$  (something that is not true of pictures in general). This constant is called the fold number. If this planar cover is made into a picture  $\mathcal{P}$ , then  $\chi(\mathcal{P})$  is the zero vector if the fold number is even, and the vector of all 1's if the fold number is odd. Thus from Theorem 4.1 we recover a result of Archdeacon and Richter that a graph is planar if and only if it has a planar cover with odd fold number [AR90]. Arkhipov's theorem can be thought of as a strengthening of Archdeacon and Richter's result that includes arbitrary pictures, not just planar covers.

To finish the section, we observe that Theorem 4.1 can be easily extended to the case that  $G$  is disconnected:

**Corollary 4.7.** Let  $(G, b)$  be a  $\mathbb{Z}_2$ -coloured graph. Then the following are equivalent:

- (a)  $J_{G,b} = 1$  in  $\Gamma(G, b)$ .
- (b)  $J_{G,b}$  is trivial in finite-dimensional representations of  $\Gamma(G, b)$ .
- (c) There is some connected component  $G'$  of  $G$  such that  $G'$  is planar and the restriction of  $b$  to  $G'$  is odd.

*Proof.* Let  $(G_1, b_1), \dots, (G_k, b_k)$  be the connected components of  $(G, b)$ . By Lemma 2.6,  $J_{G,b} = 1$  in  $\Gamma(G, b)$  if and only if  $J_{G_i, b_i} = 1$  in  $\Gamma(G_i, b_i)$  for some  $1 \leq i \leq k$ . Clearly (a) implies (b). If  $J_{G,b} \neq 1$ , then by Theorem 4.1,  $J_{G_i, b_i}$  is non-trivial in finite-dimensional representations of  $\Gamma(G_i, b_i)$  for all  $1 \leq i \leq k$ . As noted after Theorem 2.4, this means that for each  $i$ , we can find a finite-dimensional representation  $\psi_i$  of  $\Gamma(G_i, b_i)$  on  $\mathbb{C}^{n_i}$  with  $\psi_i(J_{G_i, b_i}) = -\mathbb{1}$ . Let  $m$  be the least common multiple of  $n_1, \dots, n_k$ . Then  $\psi_i^{\oplus m/n_i}$  is a representation of  $\Gamma(G_i, b_i)$  on  $\mathbb{C}^m$  sending  $J_{G_i, b_i} \mapsto -\mathbb{1}$ . By Lemma 2.6, there is a representation  $\psi$  of  $\Gamma(G, b)$  on  $\mathbb{C}^m$  sending  $J \mapsto -\mathbb{1}$ . So (a) and (b) are equivalent.

By Theorem 4.1,  $J_{G_i, b_i} = 1$  if and only if  $b_i$  is odd, and  $G_i$  is planar, so (a) and (c) are also equivalent.  $\square$

While part (c) of Corollary 4.7 is a practical criterion for testing  $J_{G,b} = 1$ , it cannot be phrased as a pattern avoidance criterion with a finite list of minors, for the same reason that  $\leq$  is not a well-quasi-order in Example 3.6.

5. EXCLUDED  $\mathbb{Z}_2$ -GRAPH MINORS FOR FINITENESS AND ABELIANNES

In this section we prove Theorems 1.4 and 1.5 by finding the excluded  $\mathbb{Z}_2$ -graph minors for finiteness and abelianness of graph incidence groups. As we will see, the proof reduces to the following statements about the graph incidence groups of uncoloured graphs:

**Proposition 5.1.**  $\Gamma(G)$  is finite if and only if  $G$  avoids  $C_2 \sqcup C_2$  and  $K_{3,6}$ .

**Proposition 5.2.**  $\Gamma(G)$  is abelian if and only if  $G$  avoids  $C_2 \sqcup C_2$  and  $K_{3,4}$ .

To explain the strategy of the proofs, we start with the following easy lemma. Recall that a cycle is a connected graph where every vertex has degree 2.

**Lemma 5.3.** If  $C$  is a cycle, then  $\Gamma(C) \cong \mathbb{Z}_2$ .

*Proof.* Suppose  $C$  has vertices  $v_i$ ,  $i \in \mathbb{Z}_n$ , where  $v_i$  is adjacent to  $v_{i-1}$  and  $v_{i+1}$  for all  $i \in \mathbb{Z}_n$ . For every  $i \in \mathbb{Z}_n$ , let  $e_i$  be the edge connecting  $v_i$  and  $v_{i+1}$ . Then  $\Gamma(C)$  is generated by  $x_{e_i}$  for  $i \in \mathbb{Z}_n$ , subject to the relations  $x_{e_i}^2 = 1$  for all  $i \in \mathbb{Z}_n$ ,  $[x_{e_{i-1}}, x_{e_i}] = 1$  for all  $i \in \mathbb{Z}_n$ , and  $x_{e_{i-1}}x_{e_i} = 1$  for all  $i \in \mathbb{Z}_n$ . These last relations imply that  $x_{e_i} = x_{e_j}$  for all  $i, j$ , so replacing all generators with a single generator  $x$ , we see that the defining presentation of  $\Gamma(C)$  is equivalent to  $\langle x : x^2 = 1 \rangle = \mathbb{Z}_2$ .  $\square$

Suppose  $G$  contains two vertex disjoint cycles, or equivalently, that  $G$  contains the disconnected union  $C_2 \sqcup C_2$  of two-cycles  $C_2$  as a graph minor. By Lemmas 3.4 and 2.8, there is a surjective homomorphism

$$\Gamma(G) \rightarrow \Gamma(C_2 \sqcup C_2) = \Gamma(C_2) * \Gamma(C_2) = \mathbb{Z}_2 * \mathbb{Z}_2.$$

The group  $\mathbb{Z}_2 * \mathbb{Z}_2$  is infinite and nonabelian, so we immediately see:

**Corollary 5.4.** If  $G$  contains  $C_2 \sqcup C_2$  as a graph minor, then  $\Gamma(G)$  is infinite and nonabelian.

So if  $\Gamma(G)$  is finite, then  $G$  cannot have two vertex disjoint cycles. Graphs without two disjoint cycles have been characterized by Lovasz [Lov65]. To state this characterization, observe that  $G$  does not have two disjoint cycles if it satisfies one of the following conditions:

- (i)  $G \setminus v$  is acyclic (and possibly empty) for some  $v \in V(G)$ ,
- (ii)  $G$  is a wheel whose spokes may be multiedges,
- (iii)  $G$  is  $K_5$ , or
- (iv)  $G$  is obtained from  $K_{3,n}$  for some  $n \geq 0$  by adding edges between vertices in the first partition. ( $K_{3,0}$  refers to the graph with 3 vertices and no edges.)

Recall that *edge subdivision* is a graph operation in which an edge  $e$  is replaced by (or *subdivided* into) two edges joined to a new vertex of degree two. A graph  $G_0$  is said to be a subdivision of  $G$  if  $G_0$  can be obtained from  $G$  by repeated edge subdivision. We consider  $G$  to be a subdivision of itself. If  $G_0$  is a subdivision of  $G$ , then every cycle of  $G_0$  is a subdivision of a cycle of  $G$ , so if  $G$  does not contain two disjoint cycles, then  $G_0$  also does not contain two disjoint cycles.

An acyclic graph, also called a forest, is a graph without cycles. We can *add a forest* to a graph  $G$  by taking the disconnected union of  $F$  and  $G$ , and then adding edges between  $F$  and  $G$  such that there is at most one edge between  $G$  and every connected component of  $F$ . The only cycles in the resulting graph  $\tilde{G}$  are the cycles of  $G$ , so if  $G$  does not contain two disjoint cycles, then neither does  $\tilde{G}$ .

Starting from a graph without two disjoint cycles, edge subdivision and adding a forest give two ways of constructing a new graph without two disjoint cycles. Lovasz's characterization states that all graphs without two disjoint cycles arise in this way from one of the graphs satisfying conditions (i)-(iv):

**Theorem 5.5** ([Lov65], see also [Bol04]). A graph  $\tilde{G}$  does not contain two vertex disjoint cycles if and only if  $\tilde{G}$  can be obtained from a graph  $G$  satisfying one of the conditions (i)-(iv) by taking a subdivision  $G_0$  of  $G$ , and then adding a (possibly empty) forest  $F$  with at most one edge between  $G_0$  and each connected component of  $F$ .

We note that the families of graphs defined by conditions (i)-(iv) are not disjoint, so Theorem 5.5 does not give a unique way of constructing every graph without two disjoint cycles. For a more precise statement where the categories do not overlap, see [Bol04, Theorem III.2.3].

It's not hard to see that subdividing and adding forests to  $G$  does not change  $\Gamma(G)$ :

**Lemma 5.6.** Let  $G_0$  be a subdivision of  $G$ . Then  $\Gamma(G_0) \cong \Gamma(G)$ .

*Proof.* Suppose  $G_1$  is the result of subdividing an edge  $e$  of  $G$  into two new edges  $e_0$  and  $e_1$ , joined by the new vertex  $v$ . In the presentation of  $\Gamma(G_1)$ , the relation for vertex  $v$  implies that  $x_{e_0} = x_{e_1}$ . Replacing  $x_{e_0}$  and  $x_{e_1}$  with  $x_e$ , we see that the presentation of  $\Gamma(G_1)$  is equivalent to the presentation of  $\Gamma(G)$ . Repeating this fact shows that  $\Gamma(G_0) \cong \Gamma(G)$  for any subdivision  $G_0$  of  $G$ .  $\square$

**Lemma 5.7.** Suppose  $\tilde{G}$  is obtained from a graph  $G$  by adding a forest  $F$  such that there is at most one edge between  $G$  and every connected component of  $F$ . Then  $\Gamma(\tilde{G}) \cong \Gamma(G)$ .

*Proof.* We can prove this by induction on the size of  $F$ . If  $F$  is empty, then the lemma is clear. If  $F$  has an isolated vertex  $v$  which is also isolated in  $\tilde{G}$ , then by Proposition 3.3, part (iii),  $\Gamma(\tilde{G}) \cong \Gamma(\tilde{G} \setminus v)$ . Suppose  $F$  is non-empty, and does not have an isolated vertex which is also isolated in  $\tilde{G}$ . If  $F$  has an isolated vertex  $v$ , then since there is at most one edge from  $v$  to  $G$  in  $\tilde{G}$ ,  $v$  must have degree one in  $\tilde{G}$ . If  $F$  does not have an isolated vertex, then every connected component of  $F$  has at least two vertices of degree one, and since at most one of these vertices can be connected to  $G$  in  $\tilde{G}$ , at least one of these vertices has degree one in  $\tilde{G}$ . Thus in both cases there is a vertex  $v$  of  $F$  such that  $v$  has degree one in  $\tilde{G}$ . Let  $e \in E(\tilde{G})$  be the edge incident to  $v$ . In the presentation of  $\Gamma(\tilde{G})$ , the relation corresponding to  $v$  is  $x_e = 1$ , so again  $\Gamma(\tilde{G}) \cong \Gamma(\tilde{G} \setminus v)$ . Now  $\tilde{G} \setminus v$  is the result of adding the forest  $F \setminus v$  to  $G$ , and since  $F \setminus v$  is smaller than  $F$ , the lemma follows by induction.  $\square$

Thus for the proofs of Propositions 5.1 and 5.2, we just need to analyze  $\Gamma(G)$  for  $G$  satisfying one of the conditions (i)-(iv) from Theorem 5.5. The graph incidence group of  $K_5$  has already been determined in Proposition 4.5. We consider each other family of graphs separately in the following subsections.

Before proceeding with the proofs of Propositions 5.1 and 5.2, we note that Lemmas 5.3 and 5.7 give a characterization of when  $\Gamma(G)$  is trivial:

**Proposition 5.8.**  $\Gamma(G)$  is trivial if and only if  $G$  is acyclic.

*Proof.* If  $G$  is acyclic, then  $G$  is the result of adding a forest to the empty graph  $G'$ . Hence  $\Gamma(G) \cong \Gamma(G') = 1$ . On the other hand, if  $G$  contains a cycle  $C$ , then  $C$  is a minor of  $G$ , and hence by Lemma 3.4 there is a surjective homomorphism  $\Gamma(G) \rightarrow \Gamma(C) = \mathbb{Z}_2$ , so  $\Gamma(G)$  is nontrivial.  $\square$

**5.1. Graphs where every cycle contains a common vertex.** In this section, we consider the first graph family listed in Theorem 5.5: graphs  $G$  for which there is a vertex  $v$  contained in all cycles, or in other words, for which  $G \setminus v$  is acyclic. An example of such a graph is shown in Figure 8.

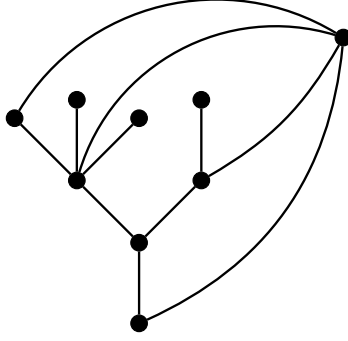


FIGURE 8. A graph where all the cycles share a common vertex.

**Proposition 5.9.** Let  $G$  be a graph for which there is a vertex  $v$  such that  $G \setminus v$  is acyclic. Then  $\Gamma(G)$  is abelian.

*Proof.* Fix some vertex  $v$  such that  $G \setminus v$  is acyclic. Note that  $G \setminus v$  is simple, i.e. there is at most one edge between every pair of vertices. Let  $K$  be the subgroup of  $\Gamma(G)$  generated by  $x_f$  for  $f \in E(v)$ , so  $K$  is abelian. We claim that  $x_e \in K$  for all  $e \in E(G \setminus v)$ , so that  $\Gamma(G) = K$ .

To prove this claim, suppose  $T$  is a connected component of  $G \setminus v$ . Pick some vertex  $w_0$  of  $T$  arbitrarily, and regard  $T$  as a rooted tree with root  $w_0$ . Every vertex  $w \neq w_0$  of  $T$  has a unique path to  $w_0$ . Following the usual conventions for rooted trees, the vertex adjacent to  $w$  in this path is called the parent of  $w$ , and the other vertices of  $T$  adjacent to  $w$  are called the descendants of  $w$ . Suppose  $w$  is some non-root vertex, and  $e$  is the edge connecting  $w$  to its parent. Let  $D \subset E(w)$  be the edges connecting  $w$  to its descendants. There might also be edges between  $w$  and  $v$  in  $G$ , so in  $\Gamma(G)$  the defining relation corresponding to vertex  $w$  states that

$$x_e = \prod_{f \in D} x_f \cdot \prod_{f' \in E(v) \cap E(w)} x_{f'}.$$

If we assume that  $x_f \in K$  for all  $f \in D$ , then  $x_e \in K$ . Thus we can use structural induction starting with the leaves of  $T$  (the vertices without any descendants) to show that  $x_e \in K$  for all  $e \in E(T)$ , proving the claim.  $\square$

Since  $\Gamma(G)$  is finitely generated by elements of order 2, if  $\Gamma(G)$  is abelian then it is a finite-dimensional  $\mathbb{Z}_2$ -vector space. When  $G \setminus v$  is acyclic for some vertex  $v$ , it is not hard to find the dimension of  $\Gamma(G)$ . Suppose  $e$  is an edge of the forest  $G \setminus v$ .



Since  $G \setminus v$  is simple, the contraction  $G/e$  is well-defined. Let  $\phi : \Gamma(G) \rightarrow \Gamma(G/e)$  be the surjective homomorphism defined in Proposition 3.3, part (ii). By the discussion after Proposition 3.3, the kernel of this homomorphism is the subgroup generated by  $[x_f, x_g]$  for  $f \in E(w_0)$  and  $g \in E(w_1)$ , where  $w_0$  and  $w_1$  are the endpoints of  $e$ . But since  $\Gamma(G)$  is abelian, this subgroup is trivial, so  $\phi$  is an isomorphism.

Since  $(G/e) \setminus v$  is also acyclic, we can continue contracting edges until we get a graph  $G'$  such that  $G' \setminus v$  has no edges. Let  $\mathbb{Z}_2^{E(G')}$  denote the free abelian group generated by the set  $\{x_e : e \in E(G')\}$ . Since  $\Gamma(G')$  is abelian,  $\Gamma(G')$  is the quotient of  $\mathbb{Z}_2^{E(G')}$  by the relations

$$(5.1) \quad \prod_{e \in E(w)} x_e = 1$$

for  $w \in V(G')$ . All edges of  $G'$  are incident to  $v$ , so the sets

$$E(w), \quad w \in V(G') \setminus \{v\}$$

partition  $E(G')$ . As a result, the relations (5.1) with  $w \in V(G') \setminus \{v\}$  involve disjoint sets of variables, and imply relation (5.1) for  $w = v$ . So  $\Gamma(G) \cong \Gamma(G') = \mathbb{Z}_2^{m-k}$ , where  $m = |E(G')|$ , and  $k$  is the number of non-isolated vertices in  $V(G') \setminus \{v\}$ . The edges of  $E(G')$  come from edges of  $G$  incident to  $v$ , and the vertices of  $V(G') \setminus \{v\}$  correspond to connected components of  $G \setminus v$ , so in terms of  $G$ ,  $m$  is the number  $|E(v)|$  of edges incident to  $v$  in  $G$ , and  $k$  is the number of connected components of  $G \setminus v$  which are connected to  $v$  in  $G$ .

**5.2. Wheel graphs with multispokes.** The simple wheel graph  $W_n$  is the graph constructed by taking a simple cycle on  $n$  vertices, adding a central vertex (for a total of  $n + 1$  vertices), and then adding an edge from each original vertex to the central vertex. The original  $n$  edges of the cycle are called the outer edges, and the  $n$  added edges connecting to the central vertex are called the spokes. The graph  $W_8$  is shown on the left in Figure 9. Clearly, simple wheels do not contain two vertex disjoint cycles since each cycle either contains the central vertex and at least two outer vertices, or is the original cycle containing all outer vertices.

**Proposition 5.10.**  $\Gamma(W_n) = \mathbb{Z}_2^n$ .

*Proof.* Let  $\{e_i : i \in \mathbb{Z}_n\}$  be the set of outer edges of  $W_n$ , and let  $\{f_i : i \in \mathbb{Z}_n\}$  be the set of spokes, where we label the two edge sets so that  $f_i$ ,  $e_{i-1}$ , and  $e_i$  are incident to a common vertex  $v_i$ . The defining relation of  $\Gamma(W_n)$  corresponding to  $v_i$  states that

$$x_{f_i} = x_{e_{i-1}}x_{e_i} = x_{e_i}x_{e_{i-1}}$$

for all  $i \in \mathbb{Z}_n$ . Because the spokes are all incident to the central vertex,

$$[x_{f_i}, x_{f_j}] = 1$$

for all  $i, j \in \mathbb{Z}_n$ . It follows that

$$(5.2) \quad x_{e_i}x_{e_j} = x_{f_{i+1}}x_{f_{i+2}} \cdots x_{f_j} = x_{f_j}x_{f_{j-1}} \cdots x_{f_{i+1}} = x_{e_j}x_{e_i}$$

for all  $i, j \in \mathbb{Z}_n$ . Since  $\Gamma(W_n)$  is generated by  $x_{e_i}$  for  $i \in \mathbb{Z}_n$ ,  $\Gamma(W_n)$  is abelian.

Now  $\Gamma(W_n)$  is abelian and generated by  $n$  elements of order 2, so  $\Gamma(W_n)$  is a  $\mathbb{Z}_2$ -vector space of dimension at most  $n$ . Given  $i \in \mathbb{Z}_n$ , let  $\bar{e}_i \in \mathbb{Z}_2^n$  denote the vector with 1 in the  $(i' + 1)$ th position and 0's in the other positions, where  $i'$  is the representative of  $i$  with  $0 \leq i' < n$ . Consider the surjective homomorphism

$\Gamma(W_n) \rightarrow \mathbb{Z}_2^n$  sending  $x_{e_i} \mapsto \bar{e}_i$ , and  $x_{f_i} \mapsto \bar{e}_{i-1} + \bar{e}_i$ . Since this homomorphism sends  $x_{f_0} \cdots x_{f_{n-1}}$  to

$$\sum_{i=0}^{n-1} \bar{e}_{i-1} + \bar{e}_i = 0$$

in  $\mathbb{Z}_2$ , this homomorphism is well-defined. So  $\Gamma(W_n)$  must have dimension  $n$ .  $\square$

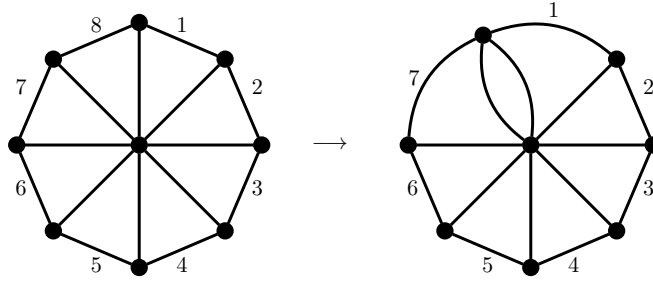


FIGURE 9. Contracting an outer edge of a simple wheel creates multiple spokes between the center and an outer vertex.

Condition (ii) from Theorem 5.5 also allows wheel graphs where the spokes may be multiedges. These graphs can be constructed by starting with  $W_n$  for some  $n$ , and adding additional edges between the outer vertices and the central vertex. Adding these edges adds additional cycles, but all these cycles still contain the central vertex and at least one outer vertex, so the resulting graphs still do not contain two disjoint cycles. As shown in Figure 9, if  $e$  is an outer edge of a simple wheel graph  $W_{n+1}$ , then the contraction  $W_{n+1}/e$  can also be obtained by adding a spoke to  $W_n$ . In general, if  $W$  is the result of adding  $k$  edges between outer vertices and the central vertex of  $W_n$ , then  $W$  can be obtained by contracting  $k$  outer edges of  $W_{n+k}$ . This can be used to determine  $\Gamma(W)$ :

**Corollary 5.11.** Let  $W$  be a wheel graph where the spokes may be multiedges. Then  $\Gamma(W) \cong \mathbb{Z}_2^m$ , where  $m$  is the number of spokes of  $W$ .

*Proof.* Suppose  $W$  is the result of adding  $k$  edges to  $W_n$ , so  $W$  has  $m = n + k$  spokes. As discussed above,  $W$  is the result of contracting  $k$  edges in  $W_{n+k}$ . By Proposition 5.10,  $\Gamma(W_{n+k})$  is abelian. As in Subsection 5.1, if  $\Gamma(G)$  is abelian, then the homomorphism  $\Gamma(G) \rightarrow \Gamma(G/e)$  from Proposition 3.3 is an isomorphism. So  $\Gamma(W) \cong \Gamma(W_{n+k}) \cong \mathbb{Z}_2^{n+k}$ .  $\square$

**5.3. Complete bipartite graphs  $K_{3,n}$ .** In this section, we consider the last family of graphs in Theorem 5.5, the graphs  $G$  which can be obtained from  $K_{3,n}$  for some  $n \geq 0$  by adding some number of edges (possibly zero) to the first partition. For this section, we refer to the two partitions of  $K_{3,n}$  as the first and second partition, with the first partition referring to the partition with 3 vertices, and the second partition referring to the partition with  $n$  vertices.

If  $n = 0$ , then the second partition is empty, and  $G$  can be any three vertex graph. We start by determining  $\Gamma(G)$  in this case.

**Proposition 5.12.** Let  $G$  be a three vertex graph. Then

$$\Gamma(G) = \begin{cases} \mathbb{Z}_2^{|E(G)|-2} & G \text{ is connected} \\ \mathbb{Z}_2^{|E(G)|-1} & G \text{ is disconnected with at least one edge} \\ \langle 1 \rangle & |E(G)| = 0 \end{cases}.$$

*Proof.* When  $G$  has three vertices, every pair of edges is incident to a common vertex, so  $\Gamma(G)$  is abelian. Thus we can think of the defining presentation of  $\Gamma(G)$  as a linear system over  $\mathbb{Z}_2$  with  $|E(G)|$  variables and three defining equations, one for each vertex. If  $G$  is connected, then the equations have a single linear dependence, so  $\Gamma(G) = \mathbb{Z}_2^{|E(G)|-2}$ . If  $G$  is disconnected and  $E(G)$  is non-empty, then  $G$  has a single isolated vertex, and  $\Gamma(G) = \mathbb{Z}_2^{|E(G)|-1}$ . If  $E(G)$  is empty (so  $G = K_{3,0}$ ) then  $\Gamma(G)$  is the trivial group.  $\square$

Moving on to the case  $n \geq 1$ , we show that edges added to the first partition of  $K_{3,n}$  end up in the centre of the solution group:

**Proposition 5.13.** Let  $G$  be a graph obtained from  $K_{3,n}$  for some  $n \geq 1$  by adding  $m$  edges to the first partition. Then  $\Gamma(G) \cong \Gamma(K_{3,n}) \times \mathbb{Z}_2^m$ .

*Proof.* Suppose  $e$  is one of the edges in  $E(G) \setminus E(K_{3,n})$ , so that the endpoints  $u$  and  $v$  of  $e$  belong to the first partition of  $K_{3,n}$ . Let  $w$  be the other vertex in the first partition of  $K_{3,n}$ . We first show that  $x_e$  is in the centre of  $\Gamma(G)$ . By definition,  $x_e$  commutes with  $x_f$  for all edges  $f$  incident to  $u$  or  $v$ , so we just need to show that  $x_e$  and  $x_f$  commute when  $f$  is not incident to  $u$  or  $v$ . If  $f$  isn't incident to  $u$  and  $v$ , then  $f$  must be incident to  $w$  and another vertex  $w'$  in the second partition of  $K_{3,n}$ . Since  $w'$  is in the second partition,  $w'$  has degree 3 in  $G$ . The other two edges  $f'$  and  $f''$  incident to  $w'$  are also incident to  $u$  and  $v$ , so  $x_e$  commutes with  $x_{f'}$  and  $x_{f''}$ . But  $x_f = x_{f'}x_{f''}$ , so  $x_e$  commutes with  $x_f$ . Thus  $x_e$  is in the centre of  $\Gamma(G)$ .

By Proposition 3.3, there is a surjective homomorphism  $\phi : \Gamma(G) \rightarrow \Gamma(G \setminus e)$  with kernel  $\langle x_e \rangle$ . Since  $x_e$  is central,  $\Gamma(G)$  is a central extension of  $\Gamma(G \setminus e)$  by  $\langle x_e \rangle$ . Pick some vertex  $w'$  from the second partition of  $K_{3,n}$ , and write  $E(w') = \{f, f', f''\}$ , where  $f, f'$ , and  $f''$  are incident to  $w, u$ , and  $v$  respectively. Define a homomorphism

$$\psi : \Gamma(G \setminus e) \rightarrow \Gamma(G) : x_{e'} \mapsto \begin{cases} x_{e'}x_e & e' \in \{f', f''\} \\ x_{e'} & \text{otherwise} \end{cases}.$$

Since  $x_e$  is central,

$$\psi \left( \prod_{e' \in E(w')} x_{e'} \right) = \psi(x_f x_{f'} x_{f''}) = x_f x_{f'} x_e x_{f''} x_e = x_f x_{f'} x_{f''} = 1,$$

while

$$\psi \left( \prod_{e' \in E(u) \setminus \{e\}} x_{e'} \right) = \psi(x_{f'}) \cdot \psi \left( \prod_{e' \in E(u) \setminus \{e, f'\}} x_{e'} \right) = \prod_{e' \in E(u)} x_{e'} = 1,$$

and the other defining relations of  $\Gamma(G \setminus e)$  can be checked similarly. So  $\psi$  is well-defined. Using the formula for  $\phi$  from Proposition 3.3, we see that  $\phi \circ \psi = \mathbb{1}_{\Gamma(G \setminus e)}$ , so  $\Gamma(G)$  is a split central extension of  $\Gamma(G \setminus e)$ , and hence  $\Gamma(G) = \Gamma(G \setminus e) \times \langle x_e \rangle$ .

$G$	$ \Gamma(G) $	$ \Gamma(G)^{Ab} $
$W_n$	$2^n$	$2^n$
$K_{3,3}$	16	16
$K_5$	64	64
$K_{3,4}$	256	64
$K_{3,5}$	8192	256
$K_{3,6}$	$\infty$	1024

TABLE 1. Order of the incidence group and its abelianization for some small graphs not containing two vertex disjoint cycles.

Finally the subgraph  $C$  of  $G$  with vertices  $u, v$ , and  $w'$ , and edges  $e, f'$ , and  $f''$  is a minor of  $G$ . The surjective homomorphism  $\Gamma(G) \rightarrow \Gamma(C)$  from Proposition 3.3 given by deleting all other vertices and edges sends  $x_e \mapsto x_e$ . Since  $x_e$  is non-trivial in  $\Gamma(C) \cong \mathbb{Z}_2$ ,  $x_e$  is non-trivial in  $\Gamma(G)$ , so  $\langle x_e \rangle \cong \mathbb{Z}_2$ . By successively deleting edges, we see that  $\Gamma(G) \cong \Gamma(K_{3,n}) \times \mathbb{Z}_2^m$ .  $\square$

Combining Propositions 5.12 and 5.13, we get the following corollary:

**Corollary 5.14.** Let  $G$  be a graph constructed by adding edges to the first partition of  $K_{3,n}$ , for some  $n \geq 0$ . Then  $\Gamma(G)$  is finite (resp. abelian) if and only if  $\Gamma(K_{3,n})$  is finite (resp. abelian).

Thus we only need to look at the groups  $K_{3,n}$  for  $n \geq 0$ . By Corollary 4.6,  $\Gamma(K_{3,3})$  is abelian, and since  $K_{3,n}$  is a minor of  $K_{3,3}$  for  $n \leq 3$ ,  $\Gamma(K_{3,n})$  is also abelian for  $n \leq 3$ . To prove Proposition 5.2, we need to show that  $\Gamma(K_{3,4})$  is nonabelian, while for Proposition 5.1, we need to show that  $\Gamma(K_{3,n})$  is finite for  $n \leq 5$  and infinite when  $n = 6$ . It's possible to determine the order of  $\Gamma(K_{3,4})$  and  $\Gamma(K_{3,5})$ , as well as the order of their abelianizations, using the GAP computer algebra package. The orders of  $\Gamma(K_{3,n})$  and  $\Gamma(K_{3,n})^{ab}$  for  $3 \leq n \leq 6$  are shown in Table 1. The order of  $\Gamma(G)$  for  $G = K_5$  and the wheel graphs  $G = W_n$  are included for comparison. It follows from this table that  $\Gamma(K_{3,4})$  is nonabelian.

We still need to show that  $\Gamma(K_{3,6})$  is infinite. Before doing this though, we give a human-readable proof that  $\Gamma(K_{3,n})$  is finite and nonabelian for  $n = 4, 5$ . Although this isn't necessary to prove Propositions 5.1 and 5.2, the proofs suggest that the structure of the groups  $\Gamma(K_{m,n})$  involve some interesting combinatorics.

To work with the groups  $\Gamma(K_{m,n})$ , we order the vertices in each partition, and label the edge from the  $i$ th vertex in the first partition to the  $j$ th vertex in the second partition by  $(j-1)m+i$ ,  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ . To visualize the presentation of  $\Gamma(K_{m,n})$ , we can draw the  $n \times m$  matrix with  $x_{(j-1)m+i}$  in the  $j$ th entry. Then  $\Gamma(K_{m,n})$  is the group generated by the entries of this matrix, such that every entry of the matrix squares to the identity, any two entries in the same row or column commute, and the product of entries in any row or column is the identity. It is also possible to get  $\Gamma(K_{m,n}, b)$  from this picture, by adding the central generator  $J$ , and setting the product of entries in any row or column to be  $J^{b(v)}$  for the corresponding vertex  $v$ , rather than the identity.

**Example 5.15.** Figure 6 shows the edge labelling above for  $K_{3,3}$ . In this figure, the first vertex partition is  $\{x, y, z\}$ , and the second partition is  $\{u, v, w\}$ , both

ordered as written. Then  $\Gamma(K_{3,3})$  is generated by the entries of

$x_1$	$x_2$	$x_3$
$x_4$	$x_5$	$x_6$
$x_7$	$x_8$	$x_9$

subject to the relations  $x_i^2 = 1$  for all  $1 \leq i \leq 9$ ,  $[x_i, x_j] = 1$  for  $(i, j)$  equal to one of the pairs  $(1, 2), (1, 3), (2, 3), (4, 5), (4, 6), (5, 6), (7, 8), (7, 9), (8, 9), (1, 4), (1, 7), (4, 7), (2, 5), (2, 8), (5, 8), (3, 6), (3, 9),$  and  $(6, 9)$ , and

$$x_1x_2x_3 = x_4x_5x_6 = x_7x_8x_9 = x_1x_4x_7 = x_2x_5x_8 = x_3x_6x_9 = 1.$$

If  $b$  is the vertex labelling of  $K_{3,3}$  with  $b(x) = b(y) = b(z) = 1$  and  $b(u) = b(v) = b(w) = 0$ , then we get  $\Gamma(K_{3,3}, b)$  by adding  $J$  to the generators along with the relations  $J^2 = [x_i, J] = 1$  for all  $1 \leq i \leq 9$ , and modifying the last group of relations to

$$x_1x_2x_3 = x_4x_5x_6 = x_7x_8x_9 = 1$$

and

$$x_1x_4x_7 = x_2x_5x_8 = x_3x_6x_9 = J.$$

Since  $K_{m,n}$  has a large number of edges, this visual representation of  $\Gamma(K_{m,n})$  and  $\Gamma(K_{m,n}, b)$  is preferable to drawing  $K_{m,n}$ . This representation is the reason that  $\mathcal{G}(K_{3,3}, b)$  is called the “magic square” game in [Mer90, Per91], and that the games  $\mathcal{G}(K_{m,n}, b)$  are called “magic rectangle” games in [AW20, AW22].

**Example 5.16.** Since  $K_{3,0}$  has no edges,  $\Gamma(K_{3,0})$  is trivial. For  $\Gamma(K_{3,1})$  and  $\Gamma(K_{3,2})$ , we can look at the matrices

$$\begin{array}{|c|c|c|} \hline x_1 & x_2 & x_3 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|} \hline x_1 & x_2 & x_3 \\ \hline x_4 & x_5 & x_6 \\ \hline \end{array}$$

respectively.  $\Gamma(K_{3,1})$  is generated by  $x_1, x_2,$  and  $x_3$ , but since the product along any column is the identity, all these generators are trivial, and hence  $\Gamma(K_{3,1})$  is trivial. Similarly,  $\Gamma(K_{3,2})$  is generated by  $x_1, \dots, x_6$ , but since  $x_1 = x_4, x_2 = x_5, x_3 = x_6,$  and  $x_1x_2x_3 = 1$ , we see that  $\Gamma(K_{3,2}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ .

To study  $\Gamma(K_{m,n})$ , it's helpful to look at the groups where we modify the matrix presentation of  $\Gamma(K_{m,n})$  by leaving out the relations stating that the product across columns is 1.

**Definition 5.17.** For any  $m, n \geq 1$ , let

$$\begin{aligned} H_{m,n} = \langle y_i, 1 \leq i \leq mn : y_i^2 = 1 \text{ for all } 1 \leq i \leq mn \\ [y_i, y_j] = 1 \text{ if } i \equiv j \pmod{m} \\ [y_{(j-1)m+i}, y_{(j-1)m+k}] = 1 \text{ for all } 1 \leq i, k \leq m, 1 \leq j \leq n \\ y_{(j-1)m+1}y_{(j-1)m+2} \cdots y_{jm} = 1 \text{ for all } 1 \leq j \leq n \rangle. \end{aligned}$$

In other words, if we fill an  $n \times m$  matrix with the indeterminates  $y_1, \dots, y_{mn}$ , reading from left to right and top to bottom, then  $H_{m,n}$  is generated by the entries of this matrix, subject to the relations stating that all entries square to the identity, that entries in the same row or column commute, and that the product of entries in any row of the matrix is the identity. We draw the matrix for  $H_{m,n}$  with a dashed bottom line, to emphasize that the column products are not included.

**Example 5.18.**  $H_{3,3}$  is generated by the entries of the matrix

$y_1$	$y_2$	$y_3$
$y_4$	$y_5$	$y_6$
$y_7$	$y_8$	$y_9$

subject to the relations  $y_i^2 = 1$  for all  $1 \leq i \leq 9$ ,  $[y_i, y_j] = 1$  for  $(i, j)$  equal to one of the pairs  $(1, 2)$ ,  $(1, 3)$ ,  $(2, 3)$ ,  $(4, 5)$ ,  $(4, 6)$ ,  $(5, 6)$ ,  $(7, 8)$ ,  $(7, 9)$ ,  $(8, 9)$ ,  $(1, 4)$ ,  $(1, 7)$ ,  $(4, 7)$ ,  $(2, 5)$ ,  $(2, 8)$ ,  $(5, 8)$ ,  $(3, 6)$ ,  $(3, 9)$ , and  $(6, 9)$ , and

$$y_1 y_2 y_3 = y_4 y_5 y_6 = y_7 y_8 y_9 = 1.$$

Something we can see very easily from the matrix presentations of  $\Gamma(K_{m,n})$  and  $H_{m,n}$  is that swapping rows and columns of the matrix gives an automorphism of the group. For instance, for  $H_{3,3}$ , swapping the first two columns of the matrix gives an automorphism  $H_{3,3} \rightarrow H_{3,3}$  sending  $y_1 \mapsto y_2$ ,  $y_2 \mapsto y_1$ ,  $y_4 \mapsto y_5$ ,  $y_5 \mapsto y_4$ ,  $y_7 \mapsto y_8$ ,  $y_8 \mapsto y_7$ , and  $y_i \mapsto y_i$  for  $i = 3, 6, 9$ . These automorphisms are very useful in analyzing these groups, as we can see in the following key example:

**Example 5.19.**  $H_{3,2}$  arises from the matrix

$y_1$	$y_2$	$y_3$
$y_4$	$y_5$	$y_6$

Unlike  $\Gamma(K_{3,2})$ , the elements  $y_1 y_4$ ,  $y_2 y_5$ , and  $y_3 y_6$  are not necessarily equal to the identity, but since elements in the same column commute, all three elements have order 2. The commutator

$$[y_1 y_4, y_2 y_5] = (y_1 y_4 y_5 y_2)^2 = (y_1 y_6 y_3 y_1)^2 = 1,$$

since  $(y_3 y_6)^2 = 1$ . Applying column swap automorphisms, we see that

$$[y_1 y_4, y_3 y_6] = [y_2 y_5, y_3 y_6] = 1$$

as well. Consider the element

$$w = (y_1 y_4)(y_2 y_5)(y_3 y_6).$$

Clearly  $w$  is invariant under the automorphism which swaps the rows of the matrix, and since the elements  $y_1 y_4$ ,  $y_2 y_5$ , and  $y_3 y_6$  pairwise commute,  $w$  is also invariant under column swaps. Now

$$w = (y_1 y_4)(y_5 y_2)(y_3 y_6) = y_1(y_4 y_5)(y_2 y_3)y_6 = y_1 y_6 y_1 y_6,$$

and since  $y_1$  and  $y_6$  commute with  $y_3$  and  $y_4$ ,

$$[w, y_3] = [w, y_4] = 1.$$

By applying row and column swaps, we see that  $w = y_i y_j y_i y_j$  for any pair of generators  $y_i$  and  $y_j$  not in the same column or row. In particular,  $[w, y_i] = 1$  for all  $i$ , so  $w$  is central.

Consider the presentation of  $\Gamma(K_{3,3}, b)$  in Example 5.15, where  $b$  is the vertex labelling of  $K_{3,3}$  where all vertices are labelled by 0 except the vertex corresponding to the bottom row. Since the generators in the first two rows of the  $3 \times 3$  matrix for  $\Gamma(K_{3,3}, b)$  satisfy the defining relations of  $H_{3,2}$ , there is a homomorphism  $H_{3,2} \rightarrow$

$\Gamma(K_{3,3}, b)$  sending  $y_i \mapsto x_i$  for all  $1 \leq i \leq 6$ . Going in the opposite direction, if we set  $J = w$ , then the entries of the  $3 \times 3$  matrix

$y_1$	$y_2$	$y_3$
$y_4$	$y_5$	$y_6$
$y_1y_4$	$y_2y_5$	$y_3y_6$

satisfy the defining relations of  $\Gamma(K_{3,3}, b)$ . So we get a homomorphism  $\Gamma(K_{3,3}, b) \rightarrow H_{3,2}$  sending  $x_i \mapsto y_i$  for all  $1 \leq i \leq 6$ ,  $x_7 \mapsto y_1y_4$ ,  $x_8 \mapsto y_2y_5$ ,  $x_9 \mapsto y_3y_6$ , and  $J \mapsto w$ . Clearly this homomorphism is an inverse to the homomorphism  $H_{3,2} \rightarrow \Gamma(K_{3,3}, b)$ , so  $H_{3,2} \cong \Gamma(K_{3,3}, b) \cong D_4 \odot D_4$ , where the central element of  $D_4 \odot D_4$  corresponds to  $w$ .

The reason Example 5.19 is so important is that in the matrix presentation of  $\Gamma(K_{3,n})$ , the entries in any pair of rows satisfy the defining relations of  $H_{3,2}$ , and thus there is a surjective homomorphism from  $H_{3,2}$  onto the subgroup generated by the entries of these rows. As a result, the entries of these rows will satisfy the identities derived in Example 5.19. For example, if we take the  $4 \times 3$  matrix

$x_1$	$x_2$	$x_3$
$x_4$	$x_5$	$x_6$
$x_7$	$x_8$	$x_9$
$x_{10}$	$x_{11}$	$x_{12}$

for  $\Gamma(K_{3,4})$ , then we see that  $x_1x_6x_1x_6$  will commute with  $x_i$  for  $1 \leq i \leq 6$ .

**Lemma 5.20.**  $\Gamma(K_{3,4})$  is finite and nonabelian.

*Proof.* As mentioned above  $x_1x_6x_1x_6$  commutes with all entries in the first two rows. In addition,

$$x_1x_6x_1x_6 = (x_1x_4)(x_2x_5)(x_3x_6) = (x_7x_{10})(x_8x_{11})(x_9x_{12}),$$

where the last identity comes from the fact that the product along any column is the identity. So  $x_1x_6x_1x_6$  also commutes with all entries in the last two rows, and hence is central. Applying row and column swap automorphisms, we see that all commutators  $[x_i, x_j]$  are central. Let  $N$  be the central (hence normal) subgroup generated by these commutators. The quotient  $\Gamma(K_{3,4})/N$  is abelian, and since  $N$  and  $\Gamma(K_{3,4})/N$  are both abelian groups finitely generated by elements of order 2, both  $N$  and  $\Gamma(K_{3,4})/N$  are finite. It follows that  $\Gamma(K_{3,4})$  is finite.

To see that  $\Gamma(K_{3,4})$  is nonabelian, note that if  $y_1, \dots, y_6$  are the generators of  $H_{3,2}$ , then the entries of the matrix

$y_1$	$y_2$	$y_3$
$y_4$	$y_5$	$y_6$
$y_1$	$y_2$	$y_3$
$y_4$	$y_5$	$y_6$

satisfy the defining relations for  $\Gamma(K_{3,4})$ . Thus there is a surjective homomorphism  $\Gamma(K_{3,4}) \rightarrow H_{3,2}$  sending

$$x_i \mapsto \begin{cases} y_i & 1 \leq i \leq 6 \\ y_{i-6} & 7 \leq i \leq 12 \end{cases}.$$

Since  $H_{3,2}$  is nonabelian, so is  $\Gamma(K_{3,4})$ . □

Since  $K_{3,4}$  is a minor of  $K_{3,n}$  for  $n \geq 4$ , Lemmas 3.4 and 5.20 imply that  $\Gamma(K_{3,n})$  is nonabelian for  $n \geq 4$ . We still need to prove:

**Lemma 5.21.**  $\Gamma(K_{3,5})$  is finite.

*Proof.* The proof is similar to the proof that  $\Gamma(K_{3,4})$  is finite, but more involved. Consider the presentation of  $\Gamma(K_{3,5})$  in terms of the matrix

$$\begin{array}{|c|c|c|} \hline x_1 & x_2 & x_3 \\ \hline x_4 & x_5 & x_6 \\ \hline x_7 & x_8 & x_9 \\ \hline x_{10} & x_{11} & x_{12} \\ \hline x_{13} & x_{14} & x_{15} \\ \hline \end{array},$$

and let

$$w = (x_{10}x_{13})(x_{11}x_{14})(x_{12}x_{15}) = (x_1x_4x_7)(x_2x_5x_8)(x_3x_6x_9).$$

From the first expression, we see that  $w$  is invariant under swapping columns, while from the second expression, we see that  $w$  is invariant under swapping any of the first three rows. Now from the second expression, we get

$$\begin{aligned} w &= x_1(x_4x_7x_5x_8)(x_2x_3)x_6x_9 \\ &= x_1(x_4x_7x_5x_8x_6x_9)(x_6x_9)x_1(x_4x_5)x_9 \\ &= x_1(x_5x_9x_5x_9)(x_6x_9x_4(x_1x_5x_9)) \\ &= (x_1x_5x_9)(x_5x_6x_4)x_1x_5x_9 \\ &= x_1x_5x_9x_1x_5x_9, \end{aligned}$$

where for the third identity, we use Example 5.19 to conclude  $x_4x_7x_5x_8x_6x_9 = x_5x_9x_5x_9$ . But since  $w$  is invariant under swapping the first and second row, and also swapping the first and second column, we get

$$x_1x_5x_9x_1x_5x_9 = w = x_5x_1x_9x_5x_1x_9.$$

Cancelling  $x_9$  on the right and rearranging the remaining terms, we see that

$$x_1x_5x_1x_5x_9 = x_9x_5x_1x_5x_1 = x_9x_1x_5x_1x_5,$$

where we use Example 5.19 again for the identity  $x_5x_1x_5x_1 = x_1x_5x_1x_5$ . Thus  $u = x_1x_5x_1x_5$  commutes with  $x_9$ . By permuting rows and columns, we see that  $u$  commutes with  $x_i$  for  $7 \leq i \leq 15$ . We also know that  $u$  commutes with  $x_i$  for  $1 \leq i \leq 6$ , so  $u$  is central. Ultimately we conclude from symmetry that commutators of the form  $[x_i, x_j]$  are central for all  $1 \leq i, j \leq 15$ . The rest of the proof is as in Lemma 5.20.  $\square$

We finish by showing that  $\Gamma(K_{3,6})$  is infinite. Recall (from, e.g. [BN98]) that a *rewriting system over a finite set  $S$*  is a finite subset  $W \subseteq S^* \times S^*$ , where  $S^*$  is the set of words over  $S$ . We write  $a \rightarrow_W b$  for words  $a, b \in S^*$  if there are words  $c, d, \ell, r \in S^*$  such that  $a = cld$ ,  $b = crd$ , and  $(\ell, r) \in W$ , and  $a \rightarrow_W^* b$  if there is a sequence  $a = a_0, a_1, \dots, a_k = b \in S^*$  with  $a_{i-1} \rightarrow_W a_i$  for all  $1 \leq i \leq k$ . We say that  $\ell \in S^*$  is a *subword* of  $a \in S^*$  if  $a = b\ell c$  for some  $b, c \in S^*$ . A word  $a \in S^*$  is a *normal form with respect to  $W$*  if  $\ell$  is not a subword of  $a$  for all pairs  $(\ell, r) \in W$ . If  $a \rightarrow_W^* b$  and  $b$  is a normal form, then  $b$  is said to be a *normal form of  $a$* . A rewriting system  $W$  is *terminating* if there is no infinite sequence



$a_1 \rightarrow_W a_2 \rightarrow_W a_3 \rightarrow_W \cdots$ , *confluent* if for every  $a, b, c \in S^*$  with  $a \rightarrow_W^* b$  and  $a \rightarrow_W^* c$ , there is  $d \in S^*$  such that  $b \rightarrow_W^* d$  and  $c \rightarrow_W^* d$ , and *locally confluent* if for every  $a, b, c \in S^*$  with  $a \rightarrow_W b$  and  $a \rightarrow_W c$ , there is  $d \in S^*$  such that  $b \rightarrow_W^* d$  and  $c \rightarrow_W^* d$ . If  $W$  is terminating, then every word  $a$  has a normal form with respect to  $W$ . If  $W$  is terminating and confluent, then every word has a unique normal form with respect to  $W$ . Newman's lemma states that if  $W$  is terminating and locally confluent, then  $W$  is confluent.

For a finite set  $S$ , an order  $\leq$  on  $S^*$  is a *reduction order* if it is a well-order (i.e. every subset of  $S^*$  has a least element) and  $a \leq b$  implies  $cad \leq cbd$  for all  $a, b, c, d \in S^*$ . If  $\leq$  is a reduction order, and  $W$  is a rewriting system such that  $\ell > r$  for all  $(\ell, r) \in W$ , then  $W$  is terminating. Suppose  $S = \{s_1, \dots, s_n\}$ , and let  $S_k = \{s_1, \dots, s_k\}$ . Define an order  $\leq_k$  on  $S_k$  inductively as follows: Let  $a <_1 b$  for  $a, b \in S_1^*$  if and only if  $b$  is longer than  $a$ . For  $k > 1$ , let  $a <_k b$  if either

- $s_k$  appears more often in  $b$  than in  $a$ , or
- $s_k$  appears  $m$  times in both  $a$  and  $b$ , and when we write  $a = a_0 s_1 a_1 s_1 \cdots s_1 a_m$ ,  $b = b_0 s_1 b_1 s_1 \cdots s_1 b_m$  for words  $a_0, \dots, a_m, b_0, \dots, b_m \in S_{k-1}^*$ , there is an index  $0 \leq i \leq m$  such that  $a_j = b_j$  for  $j < i$  and  $a_i <_{k-1} b_i$ .

The resulting order  $\leq := \leq_k$  on  $S$  is called the *wreath product order for the sequence*  $s_1, \dots, s_n$ . Wreath product orders are reduction orders.

Returning to groups, a *complete rewriting system* for a finitely presented group  $\langle S : R \rangle$  is a terminating and confluent rewriting system over  $S \cup S^{-1}$ , such that the empty word 1 is the normal form of all  $r \in R$ , and is also the normal form of  $ss^{-1}$  and  $s^{-1}s$  for all  $s \in S$ . It is well-known that if  $W$  is a complete rewriting system for  $\langle S : R \rangle$ , then the elements of  $\langle S : R \rangle$  are in bijection with the set of normal forms with respect to  $W$ .<sup>2</sup> If every element of  $S$  has order two, then it is more convenient to work with rewriting systems over  $S$  rather than  $S \cup S^{-1}$ . Specifically, if  $R$  is a set of words over a set  $S$ , then we say that a rewriting system  $W$  over  $S$  is a complete rewriting system for the group

$$\Gamma = \langle S : R \cup \{s^2 : s \in S\} \rangle$$

if  $W$  is terminating and confluent,  $r$  has normal form 1 for all relations  $r \in R$ , and  $s^2$  has normal form 1 for all  $s \in S$ . If  $W$  is such a rewriting system, then once again the elements of  $\Gamma$  are in bijection with the normal forms with respect to  $W$ .

The Knuth-Bendix algorithm is a procedure for constructing a complete rewriting system for a finitely presented group, given the group presentation and a reduction order as input. It is not guaranteed to halt, and the success and running time of the procedure is often highly dependent on the specified order. Using the implementation of Knuth-Bendix in the KBMAG package [Hol], we were able to find a complete rewriting system for the group  $H_{3,3}$ . This rewriting system is shown in Figure 10. Finding this rewriting system involved a lengthy automated search through reduction orders until we found one for which the Knuth-Bendix procedure would halt. However, it is much easier to verify that this rewriting system is complete once we've found it:

**Lemma 5.22.** The rewriting system  $W$  in Figure 10 is a complete rewriting system for  $H_{3,3}$ .

<sup>2</sup>For the reader interested in proving this themselves, note that if  $r_1 r_2^{-1} \in R$ , then  $r_1 r_2^{-1} r_2 \rightarrow_W^* r_1$  and  $r_2$  and  $r_1 r_2^{-1} r_2 \rightarrow_W^* r_1$ , so  $r_1$  and  $r_2$  must have the same normal forms.

$$\begin{aligned}
y_4y_4 &\longrightarrow 1 \\
y_7y_7 &\longrightarrow 1 \\
y_2y_2 &\longrightarrow 1 \\
y_8y_8 &\longrightarrow 1 \\
y_3y_3 &\longrightarrow 1 \\
y_6y_6 &\longrightarrow 1 \\
y_4y_7 &\longrightarrow y_7y_4 \\
y_2y_8 &\longrightarrow y_8y_2 \\
y_6y_3 &\longrightarrow y_3y_6 \\
y_2y_3 &\longrightarrow y_3y_2 \\
y_4y_6 &\longrightarrow y_6y_4 \\
y_7y_8 &\longrightarrow y_8y_7 \\
y_9 &\longrightarrow y_8y_7 \\
y_1 &\longrightarrow y_3y_2 \\
y_5 &\longrightarrow y_6y_4 \\
y_8y_3 &\longrightarrow y_3y_8y_7y_2y_7y_2 \\
y_6y_8 &\longrightarrow y_8y_7y_6y_7 \\
y_4y_3 &\longrightarrow y_3y_2y_4y_2 \\
y_7y_3 &\longrightarrow y_3y_2y_7y_2 \\
y_6y_4y_2 &\longrightarrow y_2y_6y_4 \\
y_6y_2 &\longrightarrow y_4y_2y_6y_4 \\
y_4y_8 &\longrightarrow y_8y_7y_6y_7y_6y_4 \\
y_6y_7y_6y_7 &\longrightarrow y_7y_6y_7y_6 \\
y_2y_7y_2y_7 &\longrightarrow y_7y_2y_7y_2 \\
y_4y_2y_4y_2 &\longrightarrow y_2y_4y_2y_4 \\
y_2y_7y_4y_2y_6y_7 &\longrightarrow y_6y_7y_2y_6y_7y_2y_6y_4 \\
y_2y_6y_7y_2y_6y_7 &\longrightarrow y_7y_4y_2y_6y_7y_2y_6y_4 \\
y_2y_7y_6y_7y_2y_6y_7 &\longrightarrow y_7y_2y_7y_2y_4y_2y_6y_7y_2y_6y_4 \\
y_2y_4y_2y_7y_2y_6y_7 &\longrightarrow y_4y_2y_6y_7y_4y_2y_6y_7y_2y_6y_4 \\
y_2y_4y_2y_7y_6y_7y_4y_2y_6y_7 &\longrightarrow y_4y_2y_7y_2y_6y_7y_2y_4y_2y_6y_7y_4y_2y_4 \\
y_2y_4y_2y_6y_7y_4y_2y_6y_7 &\longrightarrow y_4y_2y_7y_2y_6y_7y_2y_6y_4 \\
y_4y_2y_6y_7y_2y_7y_2y_6y_7 &\longrightarrow y_2y_7y_2y_6y_7y_2y_7y_6y_4 \\
y_2y_4y_2y_7y_4y_2y_7 &\longrightarrow y_4y_2y_7y_4y_2y_7y_2 \\
y_2y_7y_4y_2y_7y_6y_7 &\longrightarrow y_6y_7y_2y_6y_7y_2y_7y_6y_4 \\
y_2y_6y_7y_2y_7y_6y_7 &\longrightarrow y_7y_4y_2y_6y_7y_2y_7y_6y_4 \\
y_2y_4y_2y_6y_7y_2y_7 &\longrightarrow y_4y_2y_6y_7y_2y_7y_2 \\
y_2y_6y_7y_4y_2y_7 &\longrightarrow y_6y_7y_4y_2y_7y_2 \\
y_2y_7y_6y_7y_2y_7 &\longrightarrow y_6y_7y_2y_6y_7y_2y_4y_2y_6y_7y_4y_2 \\
y_4y_2y_6y_7y_2y_4y_2y_6y_7 &\longrightarrow y_2y_7y_2y_4y_2y_6y_7y_2y_4y_2y_6 \\
y_2y_4y_2y_7y_6y_7y_4y_2y_7 &\longrightarrow y_4y_2y_6y_7y_2y_7y_2y_4y_2y_6y_7y_2y_6 \\
y_4y_2y_7y_2y_6y_7y_2y_7 &\longrightarrow y_2y_6y_7y_2y_7y_2y_6y_7y_6y_4 \\
y_4y_2y_7y_4y_2y_7y_2y_6y_7 &\longrightarrow y_2y_6y_7y_4y_2y_6y_7y_2y_7y_6y_4 \\
y_4y_2y_6y_7y_4y_2y_6y_7y_2y_7 &\longrightarrow y_2y_7y_4y_2y_7y_2y_6y_7y_6y_4 \\
y_4y_2y_7y_2y_4y_2y_6y_7 &\longrightarrow y_2y_6y_7y_2y_4y_2y_6y_7y_2y_4y_2y_6 \\
y_4y_2y_6y_7y_2y_4y_2y_7y_6y_7 &\longrightarrow y_2y_7y_2y_4y_2y_6y_7y_2y_4y_2y_7y_6 \\
y_4y_2y_7y_2y_4y_2y_7y_6y_7 &\longrightarrow y_2y_6y_7y_2y_4y_2y_6y_7y_2y_4y_2y_7y_6
\end{aligned}$$

FIGURE 10. A complete rewriting system for the group  $H_{3,3}$ .

*Proof.* Let  $S = \{y_1, \dots, y_9\}$ . To show that  $W$  is terminating, we can check either by hand or on a computer that if  $(\ell, r) \in W$ , then  $\ell > r$  in the wreath product ordering for the sequence  $O = (y_4, y_6, y_2, y_7, y_5, y_8, y_3, y_9, y_1)$ . For example,  $y_1 > y_3 y_2$  because  $y_1$  occurs after  $y_2$  and  $y_3$  in the sequence  $O$ , and  $y_1$  appears fewer times in  $y_3 y_2$  than in  $y_1$ . In another example, consider the pair

$$(\ell, r) = (y_4 y_2 y_7 y_4 y_2 y_7 y_2 y_6 y_7, y_2 y_6 y_7 y_4 y_2 y_6 y_7 y_2 y_7 y_6 y_4).$$

The generator  $y_7$  occurs three times in both  $\ell$  and  $r$ , so to compare these two we look at the words  $y_4 y_2$  and  $y_2 y_6$ . Since  $y_4 > 1$ , we see that  $y_4 y_2 > y_2 y_6$ , and hence  $\ell > r$ . The other pairs can be checked similarly. Since wreath product orders are reduction orders,  $W$  is terminating.

For local confluence, it suffices to check two conditions:

- (i) If  $(ab, r_1), (bc, r_2) \in W$  for  $a, b, c, r_1, r_2 \in S^*$ , so that  $abc \rightarrow_W ar_2$  and  $abc \rightarrow_W r_1 c$ , then there is a word  $d$  such that  $ar_2 \rightarrow_W^* d$  and  $r_1 c \rightarrow_W^* d$ .
- (ii) If  $(abc, r_1), (b, r_2) \in W$ , so that  $abc \rightarrow_W r_1$  and  $abc \rightarrow_W ar_2 c$ , then there is a word  $d$  such that  $r_1 \rightarrow_W^* d$  and  $ar_2 c \rightarrow_W^* d$ .

This is time-consuming to check by hand for  $W$ , but can easily be checked on a computer.

Since  $W$  is both terminating and locally confluent, it is confluent. Clearly  $y_i^2 \rightarrow_W 1$  for all  $1 \leq i \leq 9$ . We can check either by hand or on a computer that  $r \rightarrow_W^* 1$  for all defining relations  $r$  of  $H_{3,3}$ , so  $W$  is a complete rewriting system.  $\square$

Short computer programs for performing the calculations in Lemma 5.22 can be found at [PRSS22]. The rewriting system from Figure 10 can be used to show:

**Lemma 5.23.**  $H_{3,3}$  is infinite.

*Proof.* By inspection of Figure 10, we see that  $(y_4 y_2 y_7)^n$  is a normal form for all  $n \geq 1$ . By Lemma 5.22, these elements are all distinct, so  $H_{3,3}$  is infinite.  $\square$

**Proposition 5.24.**  $\Gamma(K_{3,6})$  is infinite.

*Proof.* If  $y_1, \dots, y_9$  are the generators of  $H_{3,3}$ , then the entries of the matrix

$y_1$	$y_2$	$y_3$
$y_4$	$y_5$	$y_6$
$y_7$	$y_8$	$y_9$
$y_1$	$y_2$	$y_3$
$y_4$	$y_5$	$y_6$
$y_7$	$y_8$	$y_9$

satisfy the defining relations of  $\Gamma(K_{3,6})$ . Thus there is a surjective homomorphism  $\Gamma(K_{3,6}) \rightarrow H_{3,3}$  sending

$$x_i \mapsto \begin{cases} y_i & 1 \leq i \leq 9 \\ y_{i-9} & 10 \leq i \leq 18 \end{cases}.$$

Since  $H_{3,3}$  is infinite, so is  $\Gamma(K_{3,6})$ .  $\square$

**5.4. Proofs of characterizations for abelianness and finiteness.** We can now finish the proofs of Propositions 5.1 and 5.2.

*Proof of Proposition 5.1.* If  $G$  contains  $C_2 \sqcup C_2$  or  $K_{3,6}$  as a minor, then  $\Gamma(G)$  is infinite by Lemma 3.4, Corollary 5.4, and Proposition 5.24.

Suppose  $G$  avoids  $C_2 \sqcup C_2$  and  $K_{3,6}$ . Then  $G$  can be obtained from a graph  $G'$  satisfying conditions (i)-(iv) from Theorem 5.5, by taking a subdivision of  $G'$  and adding a forest. Since  $G'$  is a minor of  $G$ ,  $G'$  also avoids  $K_{3,6}$ . By Proposition 5.9, Corollary 5.11 and Corollary 4.6, if  $G'$  satisfies one of conditions (i)-(iii) from Theorem 5.5, then  $\Gamma(G')$  is abelian (and hence finite). Suppose  $G'$  satisfies condition (iv) from Theorem 5.5, so there is  $n \geq 0$  such that  $G'$  can be obtained from  $K_{3,n}$  by adding edges to the first partition. Since  $G'$  does not contain  $K_{3,6}$ , we must have  $n < 6$ , so  $\Gamma(K_{3,n})$  is finite by Example 5.16, Corollary 4.6, and Lemmas 5.20 and 5.21. By Corollary 5.14,  $\Gamma(G')$  is also finite. Since  $\Gamma(G) \cong \Gamma(G')$  by Lemmas 5.6 and 5.7, we conclude in all four cases that  $\Gamma(G)$  is finite.  $\square$

The proof of Proposition 5.2 is very similar:

*Proof of Proposition 5.2.* If  $G$  contains  $C_2 \sqcup C_2$  or  $K_{3,4}$  as a minor, then  $\Gamma(G)$  is nonabelian by Lemma 3.4, Corollary 5.4, and Proposition 5.20.

Suppose  $G$  avoids  $C_2 \sqcup C_2$  and  $K_{3,4}$ . Then  $G$  can be obtained from a graph  $G'$  satisfying conditions (i)-(iv) from Theorem 5.5, by taking a subdivision of  $G'$  and adding a forest. Since  $G'$  is a minor of  $G$ ,  $G'$  also avoids  $K_{3,4}$ . By Proposition 5.9, Corollary 5.11 and Corollary 4.6, if  $G'$  satisfies one of conditions (i)-(iii) from Theorem 5.5, then  $\Gamma(G')$  is abelian. Suppose  $G'$  satisfies condition (iv) from Theorem 5.5, so there is  $n \geq 0$  such that  $G'$  can be obtained from  $K_{3,n}$  by adding edges to the first partition. Since  $G'$  does not contain  $K_{3,4}$ , we must have  $n < 4$ , so  $\Gamma(K_{3,n})$  is abelian by Example 5.16 and Corollary 4.6. By Corollary 5.14,  $\Gamma(G')$  is also abelian. Since  $\Gamma(G) \cong \Gamma(G')$  by Lemmas 5.6 and 5.7, we conclude in all four cases that  $\Gamma(G)$  is abelian.  $\square$

With these propositions, we can prove Theorems 1.4 and 1.5.

*Proof of Theorem 1.4.* If  $b$  is a  $\mathbb{Z}_2$ -colouring of  $G$ , then  $\Gamma(G, b)$  is finite if and only if  $\Gamma(G)$  is finite. So the theorem follows immediately from Proposition 5.1.  $\square$

*Proof of Theorem 1.5.* Let  $(G, b)$  be a connected  $\mathbb{Z}_2$ -coloured graph, and let

$$\mathcal{F} = \{(K_{3,3}, b') : b' \text{ odd parity}\} \cup \{(K_5, b') : b' \text{ odd parity}\} \\ \cup \{(K_{3,4}, b') : b' \text{ even parity}\} \cup \{(C_2 \sqcup C_2, b') : b' \text{ any parity}\}.$$

By Lemma 3.2,  $(G, b)$  avoids  $\mathcal{F}$  if and only if either  $b$  has even parity and  $G$  avoids  $K_{3,4}$  and  $C_2 \sqcup C_2$ , or  $b$  has odd parity and  $G$  avoids  $K_{3,3}$ ,  $K_5$ , and  $C_2 \sqcup C_2$ .

If  $b$  has even parity, then  $\Gamma(G, b) \cong \Gamma(G, 0) \cong \Gamma(G) \times \mathbb{Z}_2$  by Lemma 2.5. So  $\Gamma(G, b)$  is abelian if and only if  $\Gamma(G)$  is abelian. By Proposition 5.2, this occurs if and only if  $G$  avoids  $K_{3,4}$  and  $C_2 \sqcup C_2$ .

Suppose  $b$  has odd parity. The groups  $\Gamma(K_{3,3}, b')$  and  $\Gamma(K_5, b')$  are nonabelian when  $b'$  is odd by Proposition 4.5. Since  $\Gamma(C_2 \sqcup C_2, b') / \langle J \rangle = \Gamma(C_2 \sqcup C_2)$  is nonabelian,  $\Gamma(C_2 \sqcup C_2, b')$  is nonabelian for any  $b'$ . So if  $G$  contains  $K_{3,3}$ ,  $K_5$ , or  $C_2 \sqcup C_2$ , then  $\Gamma(G, b)$  is nonabelian by Lemma 1.2. Suppose  $G$  avoids  $K_{3,3}$ ,  $K_5$ , and  $C_2 \sqcup C_2$ . Then  $G$  is planar, so  $J = 1$  in  $\Gamma(G, b)$  by Theorem 4.1. Hence

$\Gamma(G, b) \cong \Gamma(G, b)/\langle J \rangle = \Gamma(G)$ . Since  $G$  avoids  $K_{3,3}$ , it also avoids  $K_{3,4}$ , and hence  $\Gamma(G, b)$  is abelian by Proposition 5.2.

We conclude that  $\Gamma(G, b)$  is abelian if and only if  $(G, b)$  avoids  $\mathcal{F}$ .  $\square$

## 6. OPEN PROBLEMS

In Theorems 1.4 and 1.5, we characterize when  $\Gamma(G, b)$  is finite or abelian. As mentioned in the introduction, it is also interesting to ask for the forbidden minors for other quotient closed properties. Since it's connected with group stability and finite-dimensional approximations of groups (and hence with near-perfect strategies for games), amenability is a particularly interesting property to ask about:

**Problem 6.1.** Find the forbidden minors for amenability of  $\Gamma(G, b)$  and  $\Gamma(G)$ .

Since amenability is closed under extensions, and  $\mathbb{Z}_2$  and  $\mathbb{Z}$  are both amenable, the group  $\mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z}_2 \rtimes \mathbb{Z}$  is amenable. So Lovasz's characterization of graphs that avoid two disjoint cycles does not help with this problem. However,  $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 = \Gamma(C_2 \sqcup C_2 \sqcup C_2)$  is not amenable, since it contains  $\mathbb{Z} * \mathbb{Z}$  as a subgroup. Thus a starting point for this problem might be to look at graphs that avoid three disjoint cycles. We note that, like planarity testing, deciding whether a graph contains  $k$ -disjoint cycles can be done in linear time in the size of the graph [Bod94].

Another property that comes up in the study of group stability is property (T).

**Problem 6.2.** Find the forbidden graph minors characterizing property (T) for  $\Gamma(G, b)$  and  $\Gamma(G)$ .

The only groups which are both amenable and have property (T) are the finite groups, so  $\Gamma(C_2 \sqcup C_2) = \mathbb{Z}_2 * \mathbb{Z}_2$  does not have property (T). Hence if  $\Gamma(G)$  has property (T), then  $G$  does not contain two disjoint cycles. The groups (i)-(iii) in Theorem 5.5 are all finite and hence have property (T). However, we do not know whether  $\Gamma(K_{3,6})$  has property (T). If it does not, then  $\Gamma(G, b)$  and  $\Gamma(G)$  would have property (T) if and only if they are finite.

As mentioned in the introduction, while testing whether  $J = 1$  is easy for graph incidence groups, it is undecidable for solution groups. It would be interesting to know whether the word problem for graph incidence groups is decidable in general. If a group has a complete rewriting system, then its word problem is decidable, so it would be also interesting to know:

**Problem 6.3.** Is there a graph incidence group which does not have a complete rewriting system.

Doing some initial computer exploration with the KBMAG package for the GAP computer algebra system, we were able to find complete rewriting systems for the graph incidence groups  $\Gamma(G)$  of all 29 cubic graphs on at most 10 vertices, with one exception: the Petersen graph, shown in Figure 11. We were also able to find complete rewriting systems for the Petersen graph with an edge contracted or deleted. In all these cases, the Knuth-Bendix algorithm in KBMAG finished within a few seconds, and returned rewriting systems with less than 50 rules. Thus the Petersen graph might be a good candidate for a graph that does not have a complete rewriting system. Having a decidable word problem or a complete rewriting system is not a quotient property, so we do not expect these properties to be characterizable by forbidden minors.

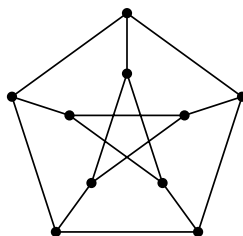


FIGURE 11. We were not able to find a complete rewriting system for the Petersen graph.

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