Antidistinguishability Conjecture

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(Based on joint work with Jamie Sikora)
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$$|\psi_?\rangle$$

Quantum states

- Represented by column vector whose indices represent classical states of that system.

- Vectors live in complex Euclidean space.

- Dirac notation is a convenient convention used in quantum information.

- "Ket" represents column vector

- "Bra" represents conjugate transpose of ket.

$$\{|\psi_1\rangle, \ldots, |\psi_n\rangle\}$$

$$|\psi_i\rangle$$

$$\langle\psi_i|.$$

Measurements

- Mechanism for extracting classical information from quantum systems.

- Collection of measurements:
    - Positive semidefinite,
    - Sum to the identity operator.

- (Born's rule): Probability of obtaining outcome "i" when measuring a quantum state.

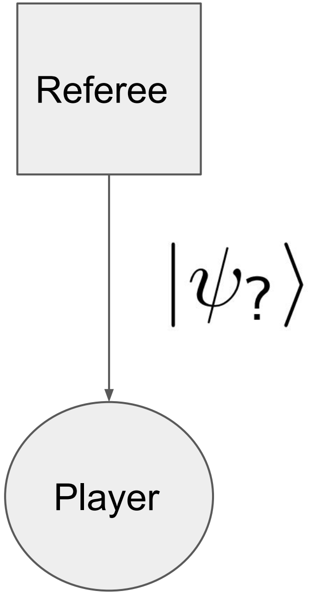
$$\{M_1, \ldots, M_n\}$$

$$p(i) = \langle \psi_i | M_i | \psi_i \rangle$$

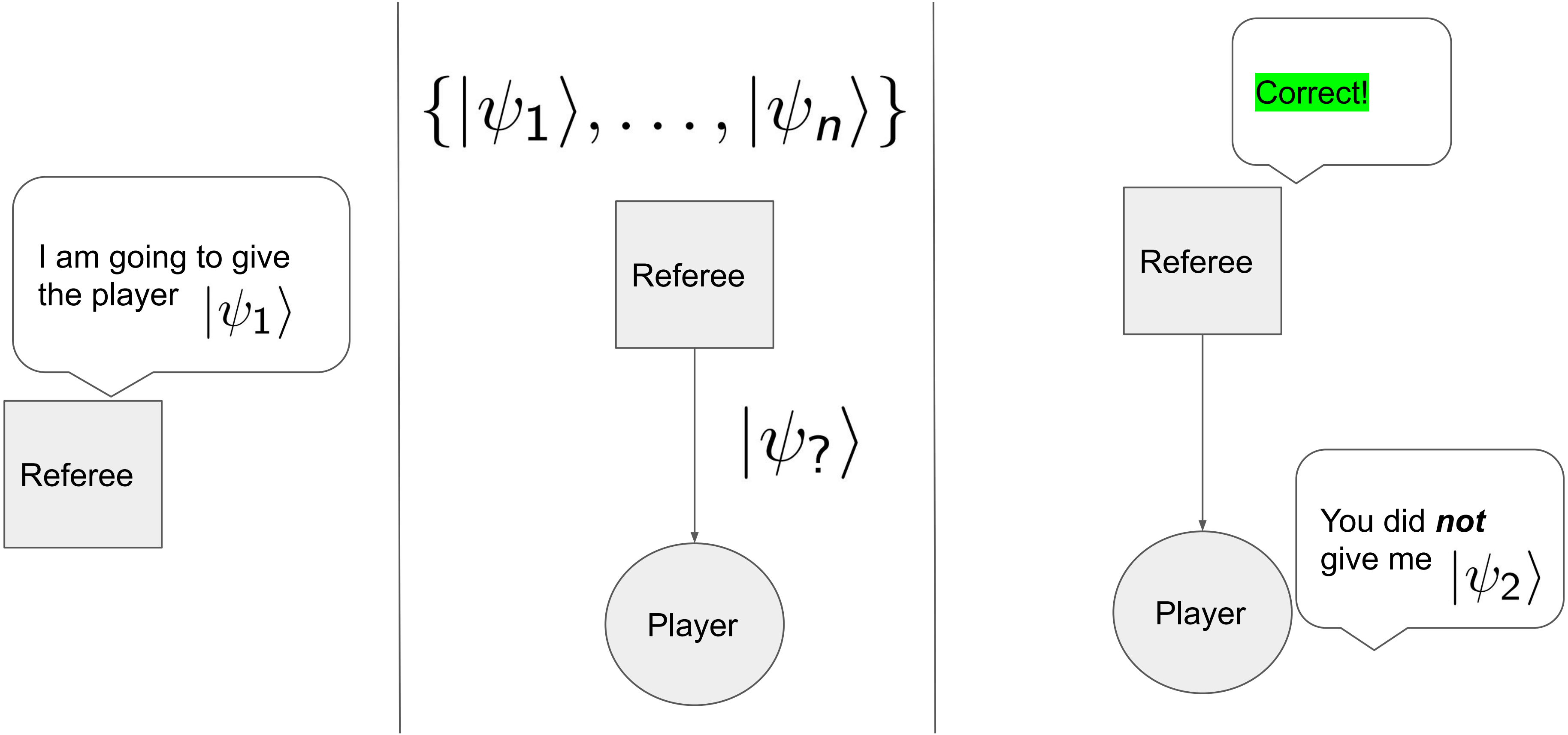
Antidistinguishability game

- Fix a set of quantum states.
- Someone hands you a state from the set at random.
- Determine which state you were ***not*** given.

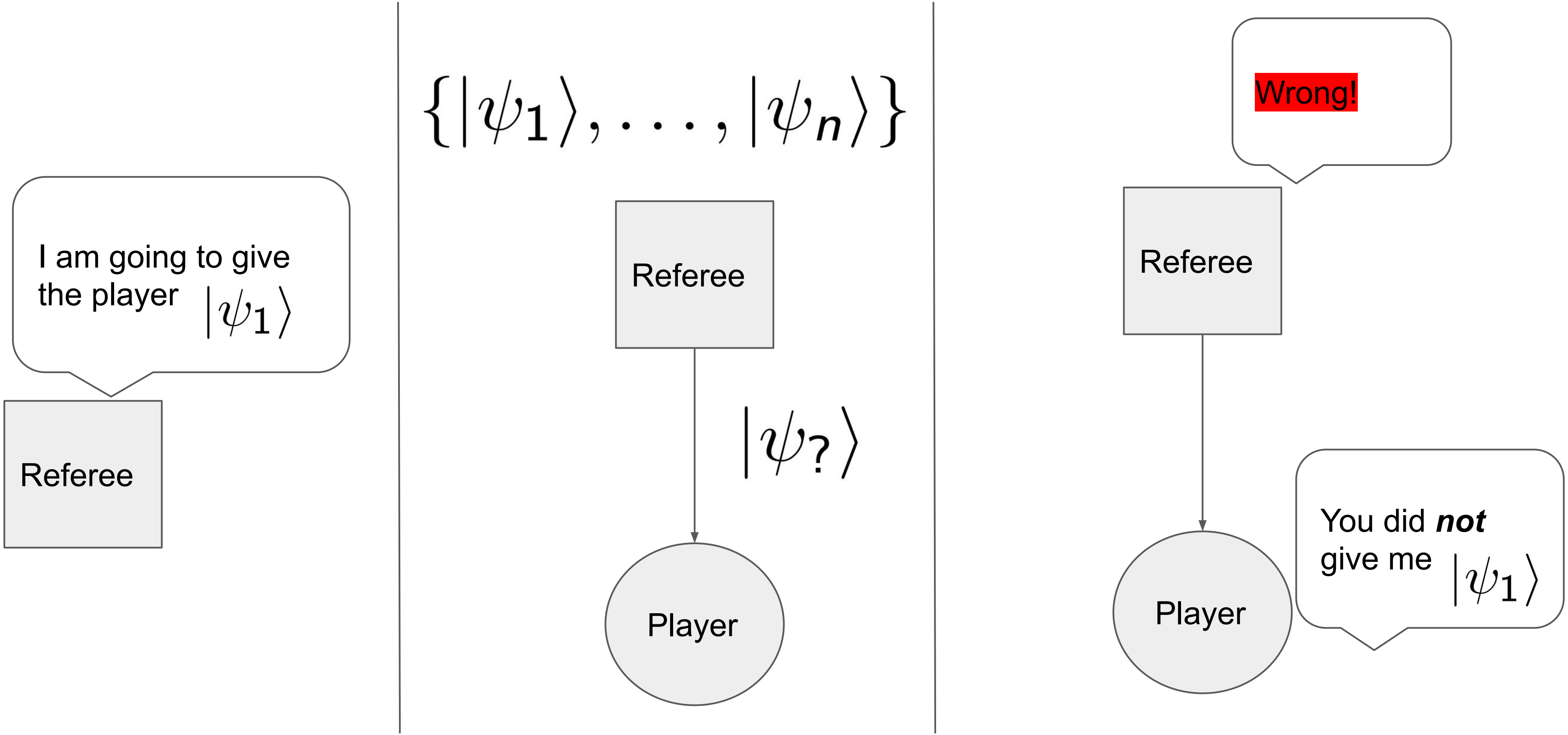
$$\{|\psi_1\rangle, \ldots, |\psi_n\rangle\}$$



Antidistinguishability game: Correct guess

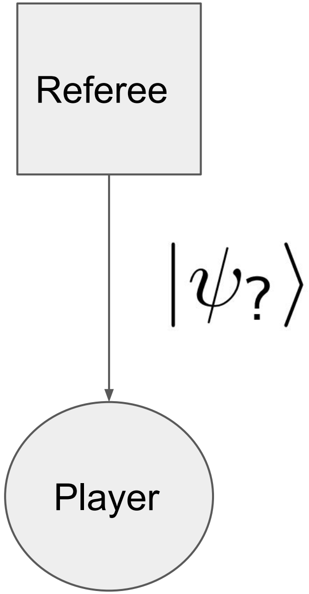


Antidistinguishability game: Incorrect guess



Antidistinguishable

$$\{|\psi_1\rangle, \ldots, |\psi_n\rangle\}$$



The set of states are **antidistinguishable** if the player can play this game perfectly.

Antidistinguishable

More formally, a set of pure quantum states

$$\{|\psi_1\rangle, \ldots, |\psi_n\rangle\} \subset \mathbb{C}^d$$

are **antidistinguishable** if there exists a set of POVMs: $\{M_1, \ldots, M_n\}$

$$\langle\psi_i|M_i|\psi_i\rangle = 0$$

for all $i \in \{1, \ldots, n\}$

Antidistinguishability applications

- Used as key part in proof of PBR theorem[1]; a result that has significance to the foundations of quantum mechanics, and more specifically, significance to how one may interpret the reality of the quantum state.

- Has been studied under the guise of *conclusive quantum state exclusion*[2] and *post-Peierls incompatibility*[3].

- A related problem (*unambiguous quantum state exclusion*) has been used to reduce the need for long-term quantum memory for digital signature schemes[4] and to develop new quantum key distribution schemes[5].

- Possible to determine whether a collection of quantum states are antidistinguishable or not based on the optimal value of a semidefinite program (SDP).

[1]Matthew F Pusey, Jonathan Barrett, and Terry Rudolph. On the reality of the quantum state. Nature Physics, 8(6):475–478, 2012.
[2]Bandyopadhyay, Somshubhro, et al., Conclusive exclusion of quantum states, Physical Review A 89.2 (2014): 022336.
[3]Caves, Carlton M., Christopher A. Fuchs, and Rüdiger Schack, Conditions for compatibility of quantum-state assignments, Physical Review A 66.6 (2002): 062111.
[4]Collins, Robert, et al., Realization of Quantum Digital Signatures without the Requirement of Quantum Memory, Physical Review Letters 113 (2014): 040502.
[5]Crickmore, Jonathan, et al. "Unambiguous quantum state elimination for qubit sequences." Physical Review Research 2.1 (2020): 013256.

Antidistinguishability conjecture[1]

A collection of pure quantum states

$$\{|\psi_1\rangle, \ldots, |\psi_d\rangle\} \subset \mathbb{C}^d$$

are **antidistinguishable** if

$$|\langle\psi_i|\psi_j\rangle| \leq (d-2)/(d-1)$$

for all $i \neq j$

[1]Vojtěch Havlíček, Jonathan Barrett, Simple communication complexity separation from quantum state antidistinguishability, Physical Review Research 2.1 (2020): 013326.

What does a validation of the conjecture imply?

If true, there exists a communication task that[1]:

- Can be solved with $\log d$ qubits

- Requires $\Omega(d \log d)$ classical bits

Would imply an **exponential separation** between classical and quantum communication complexity.

[1]Vojtěch Havlíček, Jonathan Barrett, Simple communication complexity separation from quantum state antidistinguishability, Physical Review Research 2.1 (2020): 013326.

Can we invalidate this conjecture?

- Find a collection of states that are **not** antidistinguishable but **do** satisfy the conjectured inequality.

- Need some way of determining whether an arbitrary collection of states are antidistinguishable.
    - Turns out this can be framed as a specific optimization problem.

- For d=2 and d=3, the conjecture is known to hold[1].

[1]Caves, Carlton M., Christopher A. Fuchs, and Rüdiger Schack, Conditions for compatibility of quantum-state assignments, Physical Review A 66.6 (2002): 062111.

Semidefinite programming

Primal problem

$$\begin{aligned} \text{maximize:} \quad & \langle A, X \rangle \\ \text{subject to:} \quad & \Phi(X) = B, \\ & X \in \text{Pos}(\mathcal{X}). \end{aligned}$$

Dual problem

$$\begin{aligned} \text{minimize:} \quad & \langle B, Y \rangle \\ \text{subject to:} \quad & \Phi^*(Y) \geq A, \\ & Y \in \text{Herm}(\mathcal{Y}). \end{aligned}$$

- Generalization of linear programming.
- Powerful tool with many applications in quantum information.
- SDPs are efficiently solvable (polynomial time).
- Provides an upper bound (dual) and lower bound (primal) for the problem.
- Software packages for solving SDPs exist (cvxpy, cvxopt, picos, etc.).

Semidefinite program for antidistinguishability

Whether a collection of quantum states are **antidistinguishable** can be framed as the optimal value of a **semidefinite program**[1,2].

Primal problem

$$
\begin{aligned}
\text{minimize:} \quad & \sum_{i=1}^{n} \langle \psi_i | M_i | \psi_i \rangle \\
\text{subject to:} \quad & \sum_{i=1}^{n} M_i = \mathbb{1}_{\mathcal{X}}, \\
& M_i \in \text{Pos}(\mathcal{X}), \quad \forall\, 1 \leq i \leq n.
\end{aligned}
$$

Dual problem

$$
\begin{aligned}
\text{maximize:} \quad & \text{Tr}(Y) \\
\text{subject to:} \quad & Y \preceq |\psi_i\rangle\langle\psi_i| \quad \forall\, 1 \leq i \leq n, \\
& Y \in \text{Herm}(\mathcal{Y}).
\end{aligned}
$$

Value of SDP is **zero** iff states are **antidistinguishable**.

[1] Bandyopadhyay, Somshubhro, et al., Conclusive exclusion of quantum states, Physical Review A 89.2 (2014): 022336.

[2] VR and Sikora, Jamie, A note on the inner products of pure states and their antidistinguishability, arXiv:2206.08313, 2022.

Numerical SDP solvers

- We can numerically encode and solve the antidistinguishability SDP.

- Python code that makes use of the Picos package[1] to invoke the CVXOPT solver[2].

[1]Sagnol and Stahlberg. Picos, a Python interface to conic optimization solvers. In Proceedings of the in 21st International Symposium on Mathematical Programming, 2012.
[2]Lieven Vandenberghe. The CVXOPT linear and quadratic cone program solvers. Online: http://cvxopt. org/documentation/coneprog. pdf, 2010.

Counterexample strategy

1. Generate collection of “d” random pure states of dimension “d”.

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    a. If the inequality is satisfied it implies a violation.

Counterexample strategy

1. Generate collection of “d” random pure states of dimension “d”.

2. Check whether the states are antidistinguishable (use SDP).

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    a. If the inequality is satisfied it implies a violation.

4. Repeat! Many times for d > 3.

Live Demo

https://github.com/vprusso/antidist

Counterexample for d = 4

Found example of 4 states that violate conjecture via random search.

$$|\psi_1\rangle = \begin{bmatrix} 0.50127198 - 0.037607j \\ -0.00698152 - 0.590973j \\ 0.08186514 - 0.4497548j \\ -0.01299883 + 0.43458491j \end{bmatrix}^{\mathsf{T}}, \quad |\psi_3\rangle = \begin{bmatrix} 0.31360906 + 0.46339313j \\ -0.0465825 - 0.47825017j \\ -0.10470394 - 0.11776404j \\ 0.60231515 + 0.26154959j \end{bmatrix}^{\mathsf{T}},$$

$$|\psi_2\rangle = \begin{bmatrix} -0.07115345 - 0.27080326j \\ 0.82047712 + 0.26320823j \\ 0.22105089 - 0.2091996j \\ -0.23575591 - 0.1758769j \end{bmatrix}^{\mathsf{T}}, \quad |\psi_4\rangle = \begin{bmatrix} -0.53532122 - 0.03654632j \\ 0.40955941 - 0.15150576j \\ 0.05741386 + 0.23873985j \\ -0.4737113 - 0.48652564j \end{bmatrix}^{\mathsf{T}}.$$

Other collection of 4-dimensional states were also found.

Antidistinguishability conjecture is false (d=4)

The ensemble **satisfies** the conjectured bound:

$$\max(|\langle\psi_i|\psi_j\rangle|) \approx 0.64514234... < \frac{2}{3}$$

However, the SDP tells us that these states are **not** antidistinguishable.

$$\mathrm{Tr}(Y) \approx 0.00039382039 > 0.$$

[1]Recall conjecture inequality: $|\langle\psi_i|\psi_j\rangle| \leq (d-2)/(d-1)$

“Fixing” the conjecture

Is it possible to provide a different equation to determine when an ensemble is antidistinguishable?

Corollary 3. *Let $n \geq 2$ be an integer and let $S = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle\}$. If*

$$\left|\langle\psi_i|\psi_j\rangle\right| \leq \frac{1}{\sqrt{2}}\sqrt{\frac{n-2}{n-1}} \quad \text{for all} \quad 1 \leq i \neq j \leq n$$

then S is antidistinguishable.

[1]Johnston, Nathaniel, VR, Sikora, Jamie “Antidistinguishability and Multilevel Coherence of Quantum States”, (In progress).

Conclusion

- Antidistinguishability conjecture is false for $d = 4$.
- More sophisticated methods to randomly generate non-antidistinguishable sets of states?
- Further study on properties of antidistinguishable states?
- Can anything be salvaged from a communication complexity standpoint?
- Other notions of antidistinguishability? Further applications?

Thanks!

References:
[1]Matthew F Pusey, Jonathan Barrett, and Terry Rudolph. On the reality of the quantum state. Nature Physics, 8(6):475–478, 2012.
[2]Sagnol and Stahlberg. Picos, a Python interface to conic optimization solvers. In Proceedings of the in 21st International Symposium on Mathematical Programming, 2012.
[3]Lieven Vandenberghe. The CVXOPT linear and quadratic cone program solvers. Online: http://cvxopt. org/documentation/coneprog. pdf, 2010.
[4]Vojtěch Havlíček, Jonathan Barrett, Simple communication complexity separation from quantum state antidistinguishability, Physical Review Research 2.1 (2020): 013326.
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