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SMALL SETS OF LOCALLY INDISTINGUISHABLE ORTHOGONAL MAXIMALLY ENTANGLED STATES

ALESSANDRO COSENTINO and VINCENT RUSSO
David R. Cheriton School of Computer Science and Institute for Quantum Computing
University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

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We study the problem of distinguishing quantum states using local operations and classical communication (LOCC). A question of fundamental interest is whether there exist sets of $k \leq d$ orthogonal maximally entangled states in $\mathbb{C}^d \otimes \mathbb{C}^d$ that are not perfectly distinguishable by LOCC. A recent result by Yu, Duan, and Ying [Phys. Rev. Lett. 109 020506 (2012)] gives an affirmative answer for the case $k = d$. We give, for the first time, a proof that such sets of states indeed exist even in the case $k < d$. Our result is constructive and holds for an even wider class of operations known as positive-partial-transpose measurements (PPT). The proof uses the characterization of the PPT-distinguishability problem as a semidefinite program.

Keywords: LOCC, PPT, state distinguishability, semidefinite programming
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1 Introduction

A central subject of study in quantum information theory is the interplay between entanglement and nonlocality. An important tool to study this relationship is the paradigm of local quantum operations and classical communication (LOCC). This is a subset of all global quantum operations, with a fairly intuitive physical description. In a two-party LOCC protocol, Alice and Bob can perform quantum operations only on their local subsystems and the communication must be classical. This restricted paradigm has played a crucial role in the understanding of the role of entanglement in quantum information. It has also provided a framework for the description of basic quantum tasks such as quantum key distribution and entanglement distillation.

A fundamental problem that has been studied to understand the limitations of LOCC protocols is the problem of distinguishing quantum states. The setup of the problem is pretty simple in the bipartite case. The two parties are given a single copy of a quantum state chosen with some probability from a collection of states and their goal is to identify which state was given, with the assumption that they have full knowledge of the collection. If the states are orthogonal and global operations are permitted, then it is always possible to determine the state with certainty. In contrast, if only LOCC protocols are allowed, Alice and Bob cannot in general discover the state they have been given, even if the states are orthogonal. The problem of distinguishing among a known set of orthogonal quantum states by LOCC has been studied by several researchers [1–15]. Some direct applications of this problem include secret sharing [16] and data hiding [17].

A question of basic interest is how the size of LOCC-indistinguishable sets (denoted by k in this paper) relates to the local dimension d of each of Alice's and Bob's subsystems. We know that the dimension of a quantum system puts a bound on the degree of entanglement the system could possibly have with another system. Analogously, one can ask whether the local dimension of the two subsystems plays any special role in the nonlocality exhibited by LOCC-indistinguishable sets of states.

Walgate et al. [2] proved that any two orthogonal pure states can always be perfectly distinguished by an LOCC measurement. A particularly interesting case is when the set is constituted of orthogonal states with full local rank. Regarding this case, Nathanson [8] showed that it is always possible to perfectly distinguish any three orthogonal maximally entangled states in $\mathbb{C}^3 \otimes \mathbb{C}^3$ by means of LOCC. On the other hand, it is known that $k > d$ orthogonal maximally entangled states can never be distinguished with certainty by LOCC measurements [7]. An interesting question is whether there exist sets of $k \leq d$ orthogonal maximally entangled states in $\mathbb{C}^d \otimes \mathbb{C}^d$ that are not perfectly distinguishable by LOCC, when $d > 3$. For the weaker model of *one-way* LOCC protocols, Bandyopadhyay et al. [18] showed some explicit examples of indistinguishable sets of states with the size of the sets being equal to the dimension of the subsystems, i.e., $k = d$. Recently, Yu et al. [12] gave an affirmative answer to the question for the case $k = d = 4$ in the setting of general LOCC protocols. Their result was later generalized in [14] for the case $k = d = 2^t$, where $t \geq 2$. The answer has remained elusive for the case $k < d$.

In this paper, we settle the question by exhibiting, for the first time, sets that contain fewer than d orthogonal maximally entangled states in $\mathbb{C}^d \otimes \mathbb{C}^d$, which are not perfectly distinguishable by LOCC measurements. Thus we show that the local dimension of the subsystems is not a tight bound on the size of sets of locally indistinguishable orthogonal maximally entangled states.

Even though all these results are about maximally entangled states, it should be noted that entanglement is not a necessary feature of locally indistinguishable sets of states. In a famous result, Bennett et al. [1] exhibited a set of orthogonal bipartite pure product states that are perfectly distinguishable by separable operations, but not by LOCC (see [19] for a simplified proof and a generalization of this result). In fact, if we allow states that are not maximally entangled to be in the set, we can construct indistinguishable sets with a fixed size in any dimension we like. Indeed, whenever we find a set of indistinguishable maximally entangled states for certain local dimensions, those states remain indistinguishable when embedded in any larger local dimensions. Nonetheless they are no longer maximally entangled with respect to the new larger local dimensions. On the one hand, entanglement makes distinguishability harder, but on the other hand, it can be used as a resource by the parties involved in the protocol. This makes the distinguishability problem especially interesting in the case when the set contains only maximally entangled states.

We tackle the problem by studying distinguishability of states for a class of operations broader than the class of LOCC measurements, which is the class of positive-partial-transpose (PPT) measurements. In fact, this class is even broader than the class of separable measurements, for which distinguishability of states has been studied as well [15, 20]. As opposed to the set of LOCC measurements, the set of PPT measurements has a nice mathematical structure. Moreover, optimizing over this set is a computationally easy task, whereas optimizing over the set of separable measurements is known to be an NP-hard problem [21, 22]. Several properties of PPT operations can indeed be characterized in the framework of semidefinite programming (see [23] for an example). In fact, semidefinite duality also helps to prove analytical bounds on the power of PPT operations, and therefore on the power of LOCC operations. A straightforward application of this idea is a simplified proof of the previously mentioned fact that $k > d$ orthogonal maximally entangled states cannot be perfectly distinguished by LOCC [7] (see [12] and [14] for a proof that this fact holds for PPT as well). The characterization of the PPT-distinguishability problem as a semidefinite program has been also exploited in [14] to find indistinguishable sets with size $k = d$.

A recent work by Yu et al. [13] has investigated further properties of state distinguishability by PPT. They prove a tight bound on the entanglement necessary to distinguish between three Bell states via PPT measurements. Furthermore, they show that regardless of the number of copies, a maximally entangled state cannot be distinguished from its orthogonal complement.

Before giving the definition of a PPT measurement, we review some notation. We denote by $\mathcal{A}$ and $\mathcal{B}$ the complex Euclidean spaces corresponding to Alice's and Bob's systems, respectively. We assume that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic copies of $\mathbb{C}^d$. A pure state $u \in \mathcal{A} \otimes \mathcal{B}$ is called maximally entangled if $\mathrm{Tr}_{\mathcal{A}}(uu^*) = \mathrm{Tr}_{\mathcal{B}}(uu^*) = \mathbf{1}/d$. The partial transpose is a mapping on $\mathcal{A} \otimes \mathcal{B}$ defined by tensoring the transpose mapping acting on $\mathcal{A}$ and the identity mapping acting on $\mathcal{B}$ and it is denoted as $\mathrm{T}_{\mathcal{A}} = \mathrm{T} \otimes \mathbf{1}_{\mathrm{L}(\mathcal{B})}$. Given a complex Euclidean space $\mathcal{A}$, we use the symbol $\mathrm{Herm}\,(\mathcal{A})$ to denote the set of Hermitian operators acting on $\mathcal{A}$. Let $\mathcal{A} = \mathcal{B} = \mathbb{C}^2$ and let ψ_i, for $i \in \{0, 1, 2, 3\}$, be the density operators corresponding to the standard Bell basis, that is, $\psi_i = |\psi_i\rangle \langle\psi_i|$, for $i \in \{0, 1, 2, 3\}$, where

$$|\psi_0\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\psi_1\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad |\psi_2\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}, \quad |\psi_3\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}. \tag{1}$$

Our construction is based on states that are tensor products of Bell states. We write down explicitly the action of the partial transpose on the Bell basis:

$$\mathrm{T}_{\mathcal{A}}(\psi_0) = \frac{1}{2}\mathbf{1} - \psi_2, \quad \mathrm{T}_{\mathcal{A}}(\psi_1) = \frac{1}{2}\mathbf{1} - \psi_3, \quad \mathrm{T}_{\mathcal{A}}(\psi_2) = \frac{1}{2}\mathbf{1} - \psi_0, \quad \mathrm{T}_{\mathcal{A}}(\psi_3) = \frac{1}{2}\mathbf{1} - \psi_1. \tag{2}$$

A positive operator $P \geq 0$ is called a *PPT operator* if it remains positive under the action of partial transposition, that is, $\mathrm{T}_{\mathcal{A}}(P) \geq 0$. A measurement $\{P_a \geq 0 : a \in \Gamma\}$ is called a *PPT measurement* if each measurement operator is PPT.

The maximum probability of distinguishing a set of states $\{\rho_1, \ldots, \rho_k\}$ by PPT measurements can be expressed as the optimal value of the following semidefinite program (for more details, see [14]). We are interested in perfect distinguishability, so we will assume, without loss of generality, that the states are drawn from the set with uniform probability, that is, $p_j = 1/k$, for each $j = 1, \ldots, k$.

Primal problem

$$\begin{array}{ll} \text{maximize:} & \frac{1}{k}\sum_{j=1}^{k}\langle P_j,\rho_j\rangle \\ \text{subject to:} & P_1+\cdots+P_k=\mathbf{1}_{\mathcal{A}}\otimes\mathbf{1}_{\mathcal{B}}, \\ & P_1,\ldots,P_k\geq 0, \\ & \mathrm{T}_{\mathcal{A}}(P_1),\ldots,\mathrm{T}_{\mathcal{A}}(P_k)\geq 0. \end{array} \tag{3}$$

The dual of the problem is easily obtained by routine calculation.

Dual problem

$$\begin{array}{ll} \text{minimize:} & \frac{1}{k}\,\mathrm{Tr}(Y) \\ \text{subject to:} & Y-\rho_j\geq \mathrm{T}_{\mathcal{A}}(Q_j), \quad j=1,\ldots,k\ , \\ & Y\in \mathrm{Herm}\left(\mathcal{A}\otimes\mathcal{B}\right), \\ & Q_1,\ldots,Q_k\geq 0. \end{array} \tag{4}$$

Given a set of states, an upper bound on the probability of distinguishing them by PPT measurements can be obtained by exhibiting a feasible solution of the above dual problem.

2 Main Result

For any $d\geq 4$ that is a power of 2, the following theorem shows how to construct sets of d orthogonal maximally entangled states in $\mathbb{C}^d\otimes\mathbb{C}^d$, for which the above dual problem has optimal value less than or equal to C, where $C<1$ is a constant. Given one of such sets, if we consider any of its subsets that contains only k states, then we have a set of k PPT-indistinguishable maximally entangled states in $\mathbb{C}^d\otimes\mathbb{C}^d$, where $k<d$, as long as $C<k/d$. Since any LOCC measurement is a PPT measurement, then such a set is also indistinguishable by LOCC.

Theorem 1 *For any $d=2^t$, where $t\geq 2$, it is possible to construct a set of k maximally entangled states in $\mathbb{C}^d\otimes\mathbb{C}^d$ for which there exists a feasible solution of the dual problem* (4) *with value of the objective function equal to* $(7d)/(8k)$.

Proof: For the case $t=2$ ($d=4$), a set of states was shown by Yu et al. in [12]:

$$\rho_1^{(2)}=\psi_0\otimes\psi_0,\qquad \rho_2^{(2)}=\psi_1\otimes\psi_1,\qquad \rho_3^{(2)}=\psi_2\otimes\psi_1,\qquad \rho_4^{(2)}=\psi_3\otimes\psi_1. \tag{5}$$

This being the first instance in the paper where we use Bell-diagonal states, we point out that the tensor product structure of those states should not mislead the reader when considering the cut between Alice's and Bob's systems. If we denote the local systems by $\mathcal{A}=\mathcal{A}_1\otimes\mathcal{A}_2$ and $\mathcal{B}=\mathcal{B}_1\otimes\mathcal{B}_2$, then the cut is such that the states $\rho_i^{(2)}$ lie on the space $(\mathcal{A}_1\otimes\mathcal{B}_1)\otimes(\mathcal{A}_2\otimes\mathcal{B}_2)$.

A bound of 7/8 on the optimal probability of distinguishing these states was proved in [14]. Here we write the feasible solution of the dual that achieves the value 7/8:

$$\begin{aligned}
Y^{(2)} &= \frac{1}{4}\mathbf{1}\otimes\mathbf{1} - \frac{1}{2}\,\mathrm{T}_{\mathcal{A}}(\psi_2\otimes\psi_3),\\
Q_1^{(2)} &= \frac{1}{2}[(\psi_0+\psi_1+\psi_3)\otimes\psi_2+\psi_2\otimes(\psi_0+\psi_1)],\\
Q_2^{(2)} &= \frac{1}{2}[(\psi_0+\psi_1)\otimes\psi_3+\psi_3\otimes(\psi_0+\psi_1+\psi_2)],\\
Q_3^{(2)} &= \frac{1}{2}[(\psi_1+\psi_3)\otimes\psi_3+\psi_0\otimes(\psi_0+\psi_1+\psi_2)],\\
Q_4^{(2)} &= \frac{1}{2}[(\psi_0+\psi_3)\otimes\psi_3+\psi_1\otimes(\psi_0+\psi_1+\psi_2)].
\end{aligned}$$

By using the set of equations (2), it is easy to check that the constraints of the dual problem hold for the above solution. In fact, it is a straightforward calculation to check that, for all $j\in\{1,2,3,4\}$, the following equations hold:

$$Y^{(2)}-\rho_j^{(2)}=\mathrm{T}_{\mathcal{A}}(Q_j^{(2)}). \tag{6}$$

Furthermore, we observe that $Q_1^{(2)},Q_2^{(2)},Q_3^{(2)}$, and $Q_4^{(2)}$ are positive semidefinite, and $\mathrm{Tr}(Y^{(2)})=7/2$.

For $t\geq 3$, we give a recursive construction of the states $\rho_j^{(t)}$, i.e.,

$$\rho_j^{(t)}=\begin{cases}\psi_0\otimes\rho_j^{(t-1)} & \text{if } j\leq 2^{t-1},\\ \psi_1\otimes\rho_{j-2^{t-1}}^{(t-1)} & \text{if } j>2^{t-1},\end{cases} \tag{7}$$

for $j\in\{1,\ldots,d\}$. Given this set of states, we can construct, again recursively, a feasible solution of the dual problem, which achieves the desired bound:

$$\begin{aligned}
Y^{(t)} &= (\psi_0+\psi_1)^{\otimes(t-2)}\otimes Y^{(2)},\\
Q_j^{(t)} &= (\psi_0+\psi_1)^{\otimes(t-2)}\otimes Q_r^{(2)},\quad j\in\{1,\ldots,d\},
\end{aligned} \tag{8}$$

where $r\in\{1,2,3,4\}$ so that $r-1\equiv j \pmod 4$.

We now prove that this solution satisfies the constraints of the dual problem. First, it is easy to see that $Y^{(t)}$ is Hermitian and that $Q_j^{(t)}\geq 0$, for any $j\in\{1,\ldots,d\}$. We prove by induction on t that the rest of the constraints are also satisfied, namely all the constraints of the form

$$Y^{(t)}-\rho_j^{(t)}\geq\mathrm{T}_{\mathcal{A}}(Q_j^{(t)}),\quad j\in\{1,\ldots,d\}.$$

The base case $t=2$ was considered above. By the induction hypothesis, and from the fact that $\psi_0+\psi_1\geq 0$, it holds that

$$(\psi_0+\psi_1)\otimes Y^{(t)}-(\psi_0+\psi_1)\otimes\rho_j^{(t)}\geq(\psi_0+\psi_1)\otimes\mathrm{T}_{\mathcal{A}}(Q_j^{(t)}). \tag{9}$$

From Eq. (7), we have $\rho_j^{(t+1)}=\psi_0\otimes\rho_j^{(t)}$ if $j\leq 2^t$, or $\rho_j^{(t+1)}=\psi_1\otimes\rho_{j-2^t}^{(t)}$ if $j>2^t$. Since $\psi_0,\psi_1\geq 0$, in either of the two cases we have

$$(\psi_0+\psi_1)\otimes Y^{(t)}-\rho_j^{(t+1)}\geq(\psi_0+\psi_1)\otimes\mathrm{T}_{\mathcal{A}}(Q_j^{(t)}). \tag{10}$$

From the set of equations (2), it is easy to see that

$$\mathrm{T}_{\mathcal{A}}(\psi_0 + \psi_1) = \psi_0 + \psi_1. \tag{11}$$

It follows that

$$(\psi_0 + \psi_1) \otimes Y^{(t)} - \rho_j^{(t+1)} \geq \mathrm{T}_{\mathcal{A}}[(\psi_0 + \psi_1) \otimes (Q_j^{(t)})]. \tag{12}$$

Finally, by the definition of the operators in Eq. (8), we have that

$$Y^{(t+1)} - \rho_j^{(t+1)} \geq \mathrm{T}_{\mathcal{A}}(Q_j^{(t+1)}). \tag{13}$$

In the case where we consider only k of the states we have constructed, the value of the program for this solution is equal to

$$\frac{\mathrm{Tr}(Y^{(t)})}{k} = \frac{2^{t-2}\,\mathrm{Tr}(Y^{(2)})}{k} = \frac{7d}{8k}. \tag{14}$$

This concludes the proof. □

It is possible to adapt the construction (7) and (8) in order to use a different couple of Bell states other than ψ_0 and ψ_1. However, these states are well-suited for a clearer proof, due to the Eq. (11).

Corollary 2 *For any $d = 2^t$, where $t \geq 4$, there exists a set of $k < d$ maximally entangled states in $\mathbb{C}^d \otimes \mathbb{C}^d$ that cannot be perfectly distinguished by any LOCC measurement.*

Proof: By the above Theorem, when $t \geq 4$, we can construct a set of $k < 2^t$ states that can be distinguished by any PPT measurement, and therefore any LOCC measurement, with only probability of success strictly less than 1. In fact, we have that $(7 \cdot 2^t)/(8 \cdot k) < 1$ whenever $t \geq 4$ and $k > (7 \cdot 2^t)/8$. □

Notice that the states generated by the above construction are Bell-diagonal, like the sets exhibited in [12] and [14]. A construction not based on Bell-diagonal states would be needed to generalize the result to the case when the dimension is not a power of two. Unfortunately, the most straightforward generalization, which makes use of the states corresponding to the generalized Pauli operators (see [14] for a formal definition of these states), leads to weak bounds and does not seem to give neat analytic solutions of the semidefinite program.

3 Example

As an application of our construction, we consider an example where the two parties are given a state drawn with uniform probability from the following set of $k = 15$ orthogonal maximally entangled states in $\mathbb{C}^{16} \otimes \mathbb{C}^{16}$:

$$\begin{aligned}
\rho_1 &= \psi_0 \otimes \psi_0 \otimes \psi_0 \otimes \psi_0, & \rho_2 &= \psi_0 \otimes \psi_0 \otimes \psi_1 \otimes \psi_1,\\
\rho_3 &= \psi_0 \otimes \psi_0 \otimes \psi_2 \otimes \psi_1, & \rho_4 &= \psi_0 \otimes \psi_0 \otimes \psi_3 \otimes \psi_1,\\
\rho_5 &= \psi_0 \otimes \psi_1 \otimes \psi_0 \otimes \psi_0, & \rho_6 &= \psi_0 \otimes \psi_1 \otimes \psi_1 \otimes \psi_1,\\
\rho_7 &= \psi_0 \otimes \psi_1 \otimes \psi_2 \otimes \psi_1, & \rho_8 &= \psi_0 \otimes \psi_1 \otimes \psi_3 \otimes \psi_1,\\
\rho_9 &= \psi_1 \otimes \psi_0 \otimes \psi_0 \otimes \psi_0, & \rho_{10} &= \psi_1 \otimes \psi_0 \otimes \psi_1 \otimes \psi_1,\\
\rho_{11} &= \psi_1 \otimes \psi_0 \otimes \psi_2 \otimes \psi_1, & \rho_{12} &= \psi_1 \otimes \psi_0 \otimes \psi_3 \otimes \psi_1,\\
\rho_{13} &= \psi_1 \otimes \psi_1 \otimes \psi_0 \otimes \psi_0, & \rho_{14} &= \psi_1 \otimes \psi_1 \otimes \psi_1 \otimes \psi_1,\\
\rho_{15} &= \psi_1 \otimes \psi_1 \otimes \psi_2 \otimes \psi_1.
\end{aligned}$$

The probability of distinguishing this set by any PPT measurement is less than or equal to 14/15. Examples in higher dimensions can be generated using the Python script available at [24].

It is worth noting that the "Entanglement Discrimination Catalysis" phenomenon, observed in [12] for the set (5), also applies to the set of states in the above example and to any set derived from our construction. If Alice and Bob are provided with a maximally entangled state as a resource, then they are able to distinguish the states in these sets and, when the protocol ends, they are still left with an untouched maximally entangled state. When $t = 2$, the catalyst is used to teleport the first qubit from one party to the other, say from Alice to Bob. Bob can then measure the first two qubits in the standard Bell basis and identify which of the four states was prepared. Since the third and fourth qubits are not being acted on, they can be used in a new round of the protocol. For the case $t > 2$, let us recall the recursive construction of the states $\rho_j^{(t)}$ from (7). Distinguishing between the two cases of the recursion is equivalent to distinguishing between two Bell states. And the base case is exactly the case $t = 2$ described above, with only one maximally entangled state involved in the catalysis.

4 Discussion

In this article we showed an explicit method to generate small sets of maximally entangled states that are not perfectly distinguishable by LOCC protocols. Thus we proved, for the first time, that the dimension of the local subsystems is not a tight bound on the size of sets of locally indistinguishable orthogonal maximally entangled states.

Asymptotically, our construction allows for the cardinality of these sets to be as small as $C \cdot d$, where C is a constant less than 1, and d is the dimension of each Alice's and Bob's subsystems. In particular, we have that $7/8 \le C < 1$. It is possible that this constant can be improved by using a different construction or by starting our recursive construction from a different base case. A further improvement would be to show a construction of indistinguishable sets with size $o(d)$. Another open problem is to give a more general construction that works even when d is not a power of two.

Finally, the bounds we proved in the paper hold for the class of PPT measurements. Stronger bounds might hold for the more restricted classes of LOCC or separable measurements. Navascués showed a hierarchy of semidefinite programs for the problem of state distinguishabilty by separable operations [25]. The first level of this hierarchy corresponds to the semidefinite program that we studied in this paper. An analysis of higher levels of the hierarchy may lead to stronger bounds than the one proved in this article. This idea will be developed in future work.

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