Extended nonlocal games from quantum-classical games

Theory Seminar

Vincent Russo

University of Waterloo

October 17, 2016

Outline

Extended nonlocal games and quantum-classical games

Entangled values and the dimension of entanglement

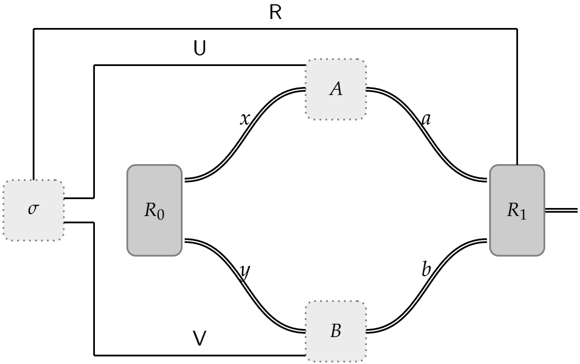
Discussion

Further work

Extended nonlocal games
and quantum-classical games

Extended nonlocal games

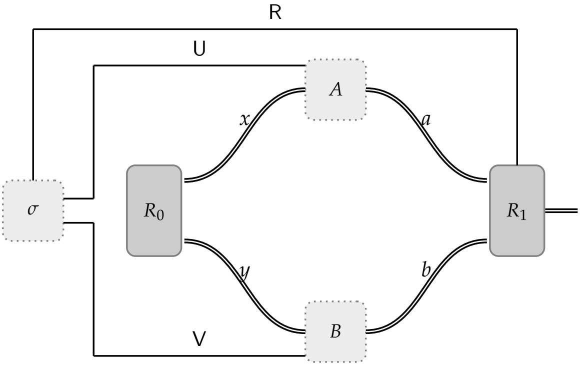
An *extended nonlocal game* (ENLG) is specified by:



- A probability distribution $\pi : X \times Y \to [0, 1]$ for alphabets X and Y.
- A collection of measurement operators $\{P_{a,b,x,y} : a \in A,\ b \in B,\ x \in X,\ y \in Y\} \subset \mathrm{Pos}(\mathcal{R})$ where $\mathcal{R}$ is the space corresponding to R and A, B are alphabets.

Extended nonlocal games

An (ENLG) is played in the following manner:



1. Alice and Bob present referee with register R.
2. Referee generates $(x, y) \in X \times Y$ according to π and sends x to Alice and y to Bob. Alice responds with a and Bob with b.
3. Referee measures R w.r.t. measurement $\{P_{a,b,x,y}, \mathbb{1} - P_{a,b,x,y}\}$. Outcome is either *loss* or *win*.

Entangled strategies for ENLGs

For an ENLG, an *entangled strategy* consists of complex Euclidean spaces $\mathcal{R}, \mathcal{U}$, and $\mathcal{V}$ as well as

- Shared state: $\sigma \in \mathrm{D}(\mathcal{U} \otimes \mathcal{R} \otimes \mathcal{V})$,
- Measurements: $\{A_a^x\} \subset \mathrm{Pos}(\mathcal{U}),\ \{B_b^y\} \subset \mathrm{Pos}(\mathcal{V})$.

Entangled strategies for ENLGs

For an ENLG, an *entangled strategy* consists of complex Euclidean spaces $\mathcal{R}, \mathcal{U}$, and $\mathcal{V}$ as well as

- Shared state: $\sigma \in \mathrm{D}(\mathcal{U} \otimes \mathcal{R} \otimes \mathcal{V})$,
- Measurements: $\{A_a^x\} \subset \mathrm{Pos}(\mathcal{U}), \ \{B_b^y\} \subset \mathrm{Pos}(\mathcal{V})$.

Winning probability for an entangled strategy is given by:

$$p = \sum_{\substack{(x,y) \in X \times Y \\ (a,b) \in A \times B}} \pi(x,y) \Big\langle A_a^x \otimes P_{a,b,x,y} \otimes B_b^y, \sigma \Big\rangle.$$

Extended nonlocal games: Winning and losing probabilities

At the end of the protocol, the referee has:

1. The state at the end of the protocol:

$$\sigma_{a,b}^{x,y} \in \mathrm{D}(\mathcal{R}).$$

2. A measurement the referee makes on its part of the state ρ:

$$P_{a,b,x,y} \in \mathrm{Pos}(\mathcal{R}).$$

The respective winning and losing probabilities are given by

$$\left\langle P_{a,b,x,y}, \sigma_{a,b}^{x,y} \right\rangle \quad \text{and} \quad \left\langle \mathbb{1} - P_{a,b,x,y}, \sigma_{a,b}^{x,y} \right\rangle.$$

Entangled values of ENLGs

For any ENLG denoted as H, the *entangled value* of H, denoted as $\omega^*(H)$, represents the supremum of the winning probabilities taken over all entangled strategies.

We may also write $\omega^*_N(H)$ to denote the *maximum* winning probability taken over all entangled strategies for which $\dim(\mathcal{U} \otimes \mathcal{V}) \leq N$, so that the entangled value of H is

$$\omega^*(H) = \lim_{N \to \infty} \omega^*_N(H).$$

ENLGs and steering

ENLGs are actually equivalent formulations of a particular type of tripartite steering.

[¶] Einstein, Podolsky, Rosen (1935): Can Quantum-Mechanical Description of Physical Reality be Considered Complete?

ENLGs and steering

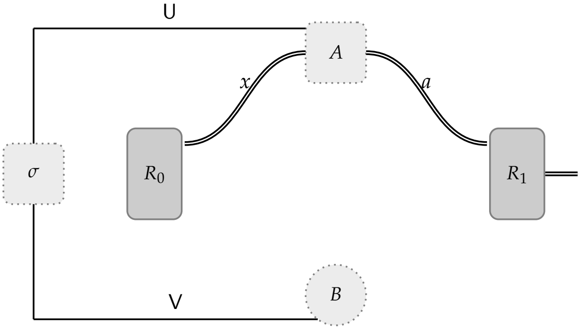
ENLGs are actually equivalent formulations of a particular type of tripartite steering.

Bipartite steering was initially introduced by Schrödinger in 1936 in an attempt to make formal the “spooky action at a distance” as discussed in the EPR paper ¶

¶ Einstein, Podolsky, Rosen (1935): Can Quantum-Mechanical Description of Physical Reality be Considered Complete?

Bipartite steering

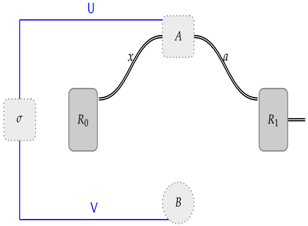
Alice and Bob each receive part of a quantum state (sent by the referee). Their goal is to determine whether this state is entangled.



- Bob's measurement device is "trusted", whereas Alice's is not:
  - Outcome of Alice's measurements are only ± 1 (a conclusive outcome) or 0 (a non-conclusive outcome).
- To demonstrate entanglement, Alice needs to "steer" Bob's state by her choice of measurement.

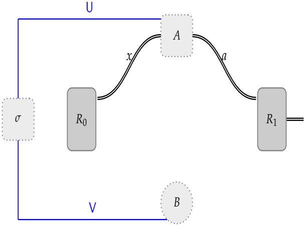
NLGs, ENLGs, and steering

Bipartite steering:

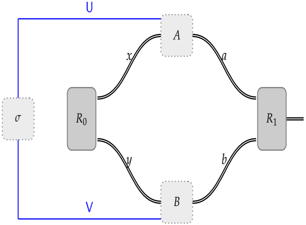


NLGs, ENLGs, and steering

Bipartite steering:

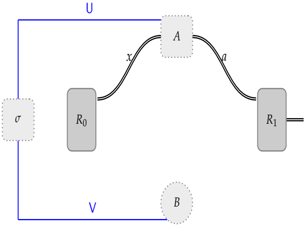


Bipartite nonlocal game:

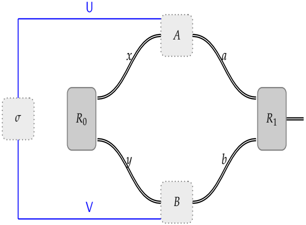


NLGs, ENLGs, and steering

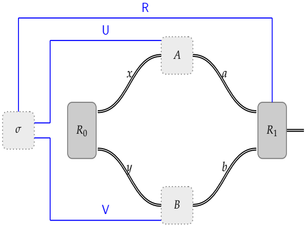
Bipartite steering:



Bipartite nonlocal game:

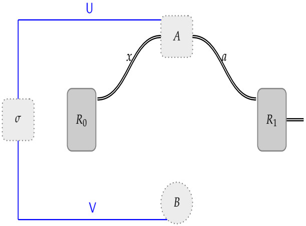


Tripartite steering with two untrusted parties (ENLG):

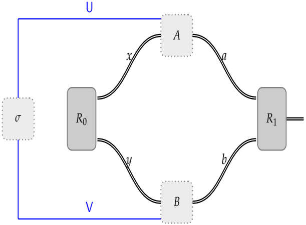


NLGs, ENLGs, and steering

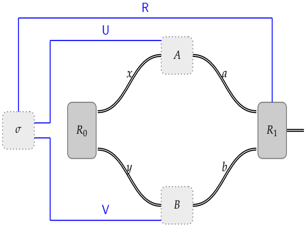
Bipartite steering:



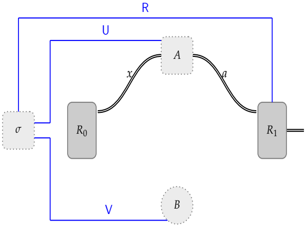
Bipartite nonlocal game:



Tripartite steering with two untrusted parties (ENLG):



Tripartite steering with one untrusted party:



ENLG and steering

Tripartite steering; same thing as before, only now we have three parties where two members are untrusted and one member is trusted.

ENLG and steering

Tripartite steering; same thing as before, only now we have three parties where two members are untrusted and one member is trusted.

In tripartite steering, Alice and Bob are the untrusted parties, and the referee is the trusted party.

ENLG and steering

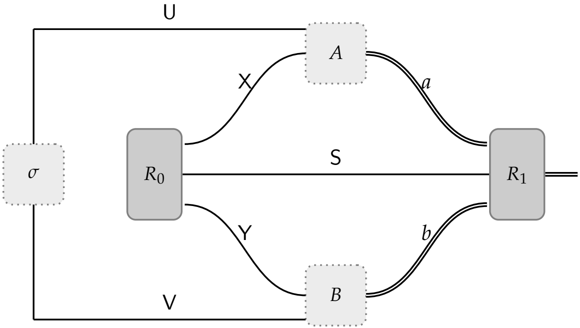
Tripartite steering; same thing as before, only now we have three parties where two members are untrusted and one member is trusted.

In tripartite steering, Alice and Bob are the untrusted parties, and the referee is the trusted party.

This isn't a talk on steering, but it's helpful to note that proving something using ENLGs will also say something about a particular type of tripartite steering.

Quantum-classical games

A *quantum-classical game* (QCG) is a cooperative game played between *Alice* and *Bob* against a *referee*.

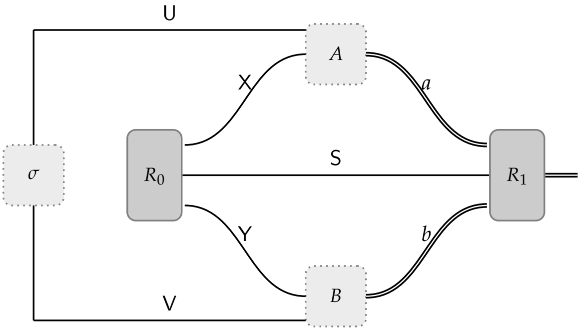


Specified by:

- A *state* $\rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{S} \otimes \mathcal{Y})$ in registers $(\mathsf{X}, \mathsf{S}, \mathsf{Y})$.
- Collection of *measurement operators* $\{Q_{a,b} : a \in A,\ b \in B\} \subset \mathrm{Pos}(\mathcal{S})$ for alphabets A and B.

Quantum-classical games

A (QCG) is played in the following manner.



1. Referee prepares $(\mathrm{X}, \mathrm{S}, \mathrm{Y})$ in state ρ and sends X to Alice and Y to Bob.
2. Alice responds with $a \in A$ and Bob with $b \in B$.
3. Referee measures S w.r.t. measurement $\{Q_{a,b}, \mathbb{1} - Q_{a,b}\}$. The outcome of this measurement results in "0" or "1", indicating a *loss* or a *win*.

Entangled strategies for QCGs

For a QCG, an *entangled strategy* consists of complex Euclidean spaces $\mathcal{U}$ and $\mathcal{V}$ as well as

- Shared state: $\sigma \in \mathrm{D}(\mathcal{U} \otimes \mathcal{V})$,
- Measurements: $\{A_a : a \in A\} \subset \mathrm{Pos}(\mathcal{U} \otimes \mathcal{X}), \{B_b : b \in B\} \subset \mathrm{Pos}(\mathcal{V} \otimes \mathcal{Y})$.

Entangled strategies for QCGs

For a QCG, an *entangled strategy* consists of complex Euclidean spaces $\mathcal{U}$ and $\mathcal{V}$ as well as

- Shared state: $\sigma \in \mathrm{D}(\mathcal{U} \otimes \mathcal{V})$,
- Measurements: $\{A_a : a \in A\} \subset \mathrm{Pos}(\mathcal{U} \otimes \mathcal{X}), \{B_b : b \in B\} \subset \mathrm{Pos}(\mathcal{V} \otimes \mathcal{Y})$.

Winning probability for a entangled strategy is given by:

$$p = \sum_{(a,b) \in A \times B} \left\langle A_a \otimes Q_{a,b} \otimes B_b, W \left(\sigma \otimes \rho \right) W^* \right\rangle,$$

where W is the unitary operator that corresponds to the natural re-ordering of registers consistent with the tensor product operators.

Entangled values for QCGs

For any QCG denoted as G, the *entangled value* of G, denoted as $\omega^*(G)$, represents the supremum of the winning probabilities taken over all entangled strategies.

We may also write $\omega^*_N(G)$ to denote the *maximum* winning probability taken over all entangled strategies for which $\dim(\mathcal{U} \otimes \mathcal{V}) \leq N$, so that the entangled value of G is

$$\omega^*(G) = \lim_{N\to\infty} \omega^*_N(G).$$

Entangled values and the dimension of entanglement

Values and the dimension of shared entanglement

Question: Does the dimensionality of the state that Alice and Bob share determine how well Alice and Bob perform?

¶ Regev, Vidick (2012): Quantum XOR games

Values and the dimension of shared entanglement

Question: Does the dimensionality of the state that Alice and Bob share determine how well Alice and Bob perform?

Partial answer: In [Regev,Vidick (2012)] ¶, the authors showed that there exists a specific class of QCG such that if the dimension of Alice and Bob's quantum system, N, is finite then $\omega_N^*(G) < 1$, but $\omega^*(G) = 1$.

¶ Regev, Vidick (2012): Quantum XOR games

Values and the dimension of shared entanglement

Question: Does the dimensionality of the state that Alice and Bob share determine how well Alice and Bob perform?

Partial answer: In [Regev,Vidick (2012)] ¶, the authors showed that there exists a specific class of QCG such that if the dimension of Alice and Bob's quantum system, N, is finite then $\omega_N^*(G) < 1$, but $\omega^*(G) = 1$.

What about ENLG?

¶ Regev, Vidick (2012): Quantum XOR games

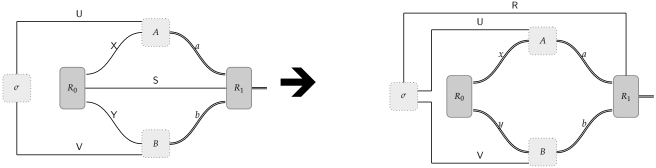
Relationship between ENLGs and QCGs

Main question: Does there also exist an ENLG, H, such that $\omega^*(H) = 1$ and $\omega^*_N(H) < 1$ when N is finite?

Relationship between ENLGs and QCGs

Main question: Does there also exist an ENLG, H, such that $\omega^*(H) = 1$ and $\omega^*_N(H) < 1$ when N is finite?

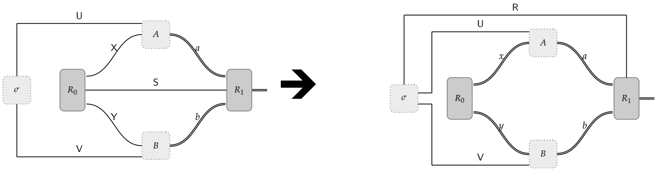
- It is possible to construct an ENLG from any QCG (not obvious).



Relationship between ENLGs and QCGs

Main question: Does there also exist an ENLG, H, such that $\omega^*(H) = 1$ and $\omega^*_N(H) < 1$ when N is finite?

- It is possible to construct an ENLG from any QCG (not obvious).



- From this construction, it turns out that this property also holds for ENLG, that is, there does exist an ENLG such that Alice and Bob can only win with certainty iff they share an infinite-dimensional state.

Constructing ENLGs from QCGs

General idea: Given a strategy for a QCG, G, show that it's possible to adapt this strategy for an ENLG, H, and vice-versa.

Approach:

- Show that for an arbitrary and fixed strategy for G, that it's possible to adapt this strategy for H.
- Show that Alice and Bob cannot do any better.

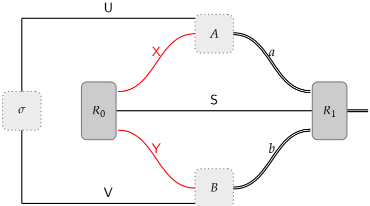
Key idea (for one direction of proof)

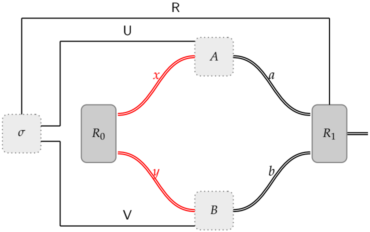
Show that for an arbitrary and fixed strategy for G, that it's possible to adapt this strategy for H.

Key idea (for one direction of proof)

Show that for an arbitrary and fixed strategy for G, that it's possible to adapt this strategy for H.

Main restriction: In G, the referee is sending quantum registers, but in H, the referee is restricted to sending classical questions.

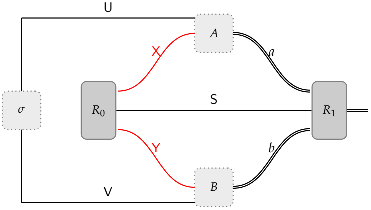


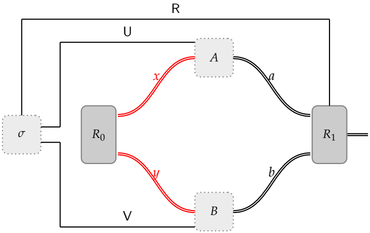


Key idea (for one direction of proof)

Show that for an arbitrary and fixed strategy for G, that it's possible to adapt this strategy for H.

Main restriction: In G, the referee is sending quantum registers, but in H, the referee is restricted to sending classical questions.





Key idea: Use teleportation to transmit X and Y in game H.

Attempt 1: Adapting via teleportation

Protocol:

1. Alice and Bob prepare $\rho \in \mathrm{D}(\mathcal{U} \otimes (\mathcal{X} \otimes \mathcal{Y}) \otimes \mathcal{V})$ in registers $(\mathsf{U}, \mathsf{X}, \mathsf{Y}, \mathsf{V})$ such that Alice/Ref and Bob/Ref share pairs of maximally entangled states.
2. Referee desires to transmit states that he creates held in X' and Y' to Alice and Bob. To do so, he measures $(\mathsf{X}, \mathsf{X}')$ and $(\mathsf{Y}, \mathsf{Y}')$ in the Bell basis to generate and send (x, y) to Alice and Bob.
3. Alice and Bob complete the teleportation protocol by applying appropriate unitaries to their system based on (x, y).

Attempt 1: Adapting via teleportation

Protocol:

1. Alice and Bob prepare $\rho \in \mathrm{D}(\mathcal{U} \otimes (\mathcal{X} \otimes \mathcal{Y}) \otimes \mathcal{V})$ in registers $(\mathsf{U}, \mathsf{X}, \mathsf{Y}, \mathsf{V})$ such that Alice/Ref and Bob/Ref share pairs of maximally entangled states.
2. Referee desires to transmit states that he creates held in X' and Y' to Alice and Bob. To do so, he measures $(\mathsf{X}, \mathsf{X}')$ and $(\mathsf{Y}, \mathsf{Y}')$ in the Bell basis to generate and send (x, y) to Alice and Bob.
3. Alice and Bob complete the teleportation protocol by applying appropriate unitaries to their system based on (x, y).

Problem: The definition of an ENLG requires that questions (x, y) are generated randomly.

Attempt 2: Post-selected teleportation protocol for H

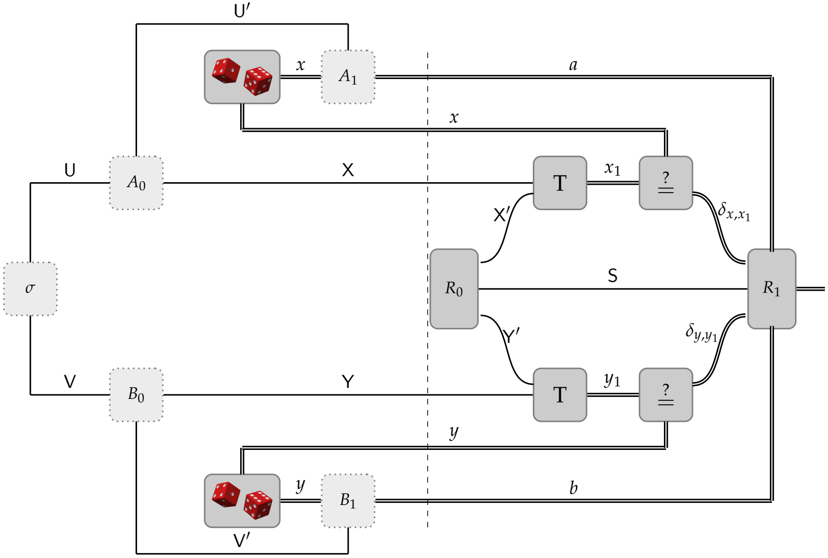
Problem: Vanilla teleportation is not enough (the questions (x, y) need to be generated *independent* of the state of registers (X, Y)).

Attempt 2: Post-selected teleportation protocol for H

Problem: Vanilla teleportation is not enough (the questions (x, y) need to be generated *independent* of the state of registers (X, Y)).

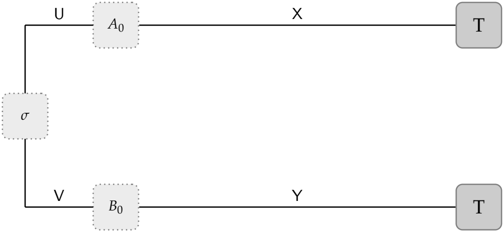
Idea: Let (x, y) be selected at random, but then compare (x, y) to hypothetical measurement results that *would* be obtained if the referee *were* to perform teleportation.

Post-selected teleportation protocol for H



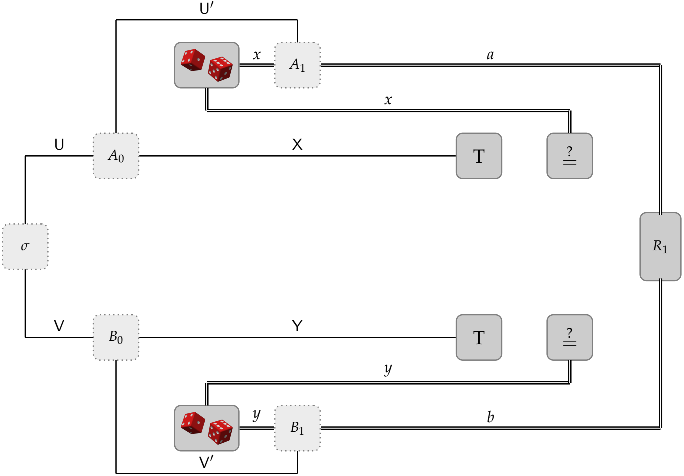
Step 1: Post-selected teleportation protocol for H

Alice and Bob prepare $\sigma \in \mathrm{D}(\mathcal{U} \otimes (\mathcal{X} \otimes \mathcal{Y}) \otimes \mathcal{V})$.



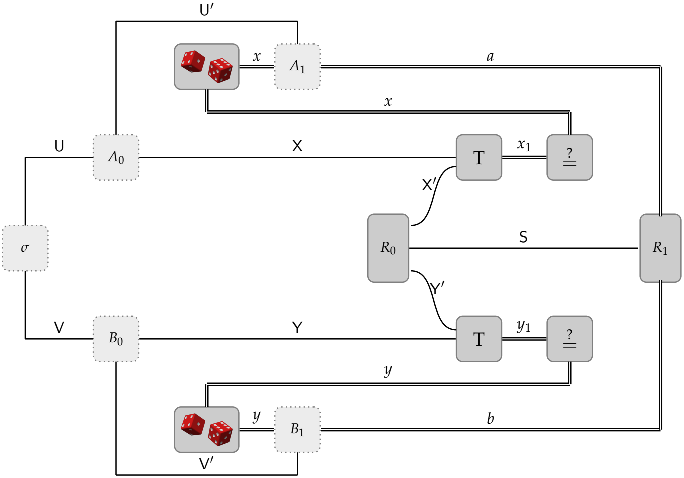
Step 2: Post-selected teleportation protocol for H

Referee randomly selects and sends (x, y); keeps a local copy. Alice and Bob respond with (a, b).



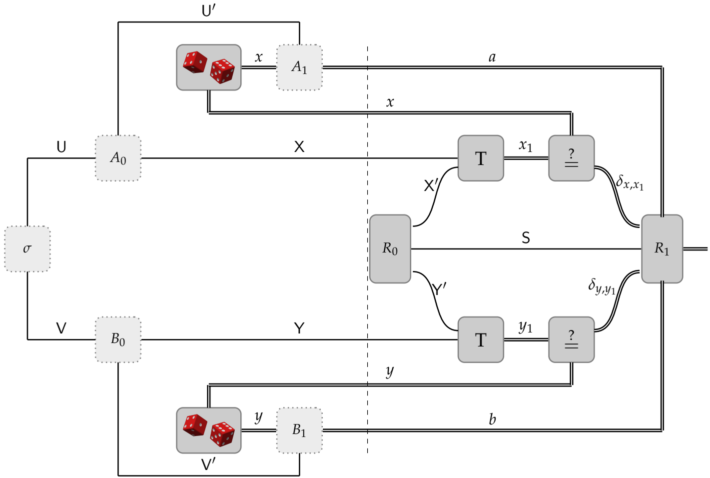
Step 3: Post-selected teleportation protocol for H

Referee prepares $\rho \in \mathrm{D}(\mathcal{X}' \otimes \mathcal{S} \otimes \mathcal{Y}')$. Performs teleportation using $(\mathsf{X}, \mathsf{X}')$ and $(\mathsf{Y}, \mathsf{Y}')$ resulting in outcomes (x_1, y_1).



Step 4: Post-selected teleportation protocol for H

1. If $x \neq x_1$ or $y \neq y_1$: teleportation fails; Alice and Bob win.
2. If $x = x_1$ and $y = y_1$: teleportation succeeds; Referee measures w.r.t. $\{P_{a,b,x,y}, \mathbb{1} - P_{a,b,x,y}\}$.



Discussion

Similar result for nonlocal games?

- Our result: There exists some ENLG such that $\omega_N^*(H) < 1$ and $\omega^*(H) = 1$. That is, for certain ENLG, increasing amounts of entanglement yield higher winning probabilities.

Similar result for nonlocal games?

- Our result: There exists some ENLG such that $\omega_N^*(H) < 1$ and $\omega^*(H) = 1$. That is, for certain ENLG, increasing amounts of entanglement yield higher winning probabilities.

- Known result: There exists some QCG such that $\omega_N^*(G) < 1$ and $\omega^*(G) = 1$. That is, for certain QCG, increasing amounts of entanglement yield higher winning probabilities.

Similar result for nonlocal games?

- Our result: There exists some ENLG such that $\omega_N^*(H) < 1$ and $\omega^*(H) = 1$. That is, for certain ENLG, increasing amounts of entanglement yield higher winning probabilities.

- Known result: There exists some QCG such that $\omega_N^*(G) < 1$ and $\omega^*(G) = 1$. That is, for certain QCG, increasing amounts of entanglement yield higher winning probabilities.

- Unknown: Does there exist some NLG, G, with similar properties, that is $\omega_N^*(G) < 1$ and $\omega^*(G) = 1$?

ENLGs and tripartite steering

Recall, ENLG may be viewed equivalently as a tripartite steering scenario.

ENLGs and tripartite steering

Recall, ENLG may be viewed equivalently as a tripartite steering scenario.

Proving results about ENLGs gives us corresponding results about tripartite steering. What does our result imply in the context of steering?

ENLGs and tripartite steering

Recall, ENLG may be viewed equivalently as a tripartite steering scenario.

Proving results about ENLGs gives us corresponding results about tripartite steering. What does our result imply in the context of steering?

Result: Our result implies the existence of a tripartite steering inequality that is maximally violated only by a quantum state with dimension approaching infinity. For any finite-dimensional state, this steering inequality cannot be maximally violated.

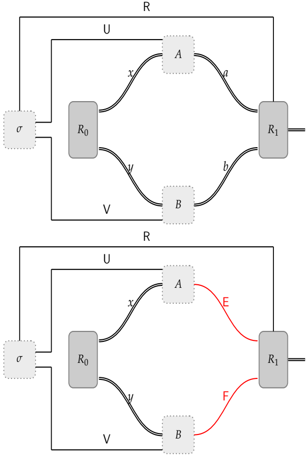
Further work

Open questions

- (Hard problem): Does there exist any nonlocal game, G, such that $\omega^*(G) = 1$ and $\omega^*_N(G) < 1$ for all N?
- (More general): Can the study of extended nonlocal games reveal anything further about the properties of tripartite steering?

A related question: Swapping rounds of communication

How powerful is the extended nonlocal game model? What happens when you substitute classical for quantum rounds of communication or vice-versa?



Thanks

Thanks for listening!

This work is primarily based on:

- N. Johnston, R. Mittal, V. R., J. Watrous.
  Extended nonlocal games and monogamy-of-entanglement games.
  Proc. R. Soc. A 472:20160003, 2016.
- V. R., J. Watrous.
  Extended nonlocal games from quantum-classical games.
  In preparation.