# Extended nonlocal games and monogamy-of-entanglement games

## Theory Seminar

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# Outline

Nonlocal games

Extended nonlocal games

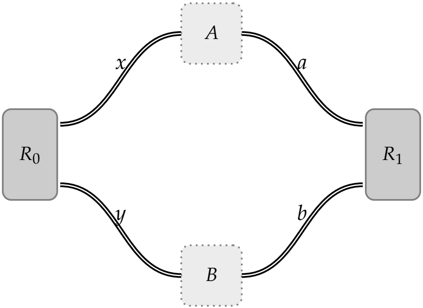
Monogamy-of-entanglement games

Open questions

# Nonlocal games

# Nonlocal games

A *nonlocal game* is a cooperative game played between *Alice* and *Bob* against a *referee*.




1. Question and answer sets: $(\Sigma_A, \Sigma_B)$ and $(\Gamma_A, \Gamma_B)$,
2. Distributions on question pairs: $\pi : \Sigma_A \times \Sigma_B \to [0, 1]$ ,
3. A predicate $V : \Gamma_A \times \Gamma_B \times \Sigma_A \times \Sigma_B \to \{0, 1\}$, where

$$V(a, b|x, y) = \begin{cases} 1 & \text{if Alice and Bob win}, \\ 0 & \text{if Alice and Bob lose}. \end{cases}$$

# Strategies and values for nonlocal games

Alice and Bob could use different types of *strategies*:

- *Classical strategies:* Alice and Bob answer deterministically, determined by functions of $f : \Sigma_A \to \Gamma_A$ and $g : \Sigma_B \to \Gamma_B$.

- *Quantum strategies:* Alice and Bob share a joint quantum system $\rho \in \mathrm{D}(\mathcal{A} \otimes \mathcal{B})$ and allow their answers to be outcomes of measurements on this shared system.

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The *value* of a nonlocal game is the maximal winning probability for the players to win over all strategies of a specified type.

For a nonlocal game, $G$, we denote the classical and quantum values as

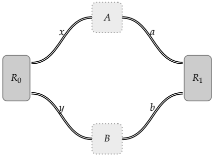
- Classical value: $\omega(G)$,
- Quantum value: $\omega^*(G)$.

# Example: The CHSH game

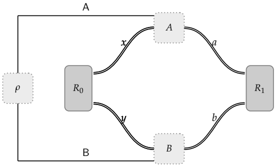
The CHSH game ($G_{\mathsf{CHSH}}$). Question and answer sets over $\{0,1\}$.
Question pairs $\{00, 01, 10, 11\}$ selected with equal probability.
Winning condition iff $a \oplus b = x \wedge y$.

$$\omega(G_{\mathsf{CHSH}}) < \omega^*(G_{\mathsf{CHSH}})$$

- $\omega(G_{\mathsf{CHSH}}) = \frac{3}{4} = 0.75$:




- $\omega^*(G_{\mathsf{CHSH}}) = \cos^2(\frac{\pi}{8}) \approx 0.8536$:



# Demo Time: CHSH game in QETLAB

CHSH_GAME.M

# Extended nonlocal games

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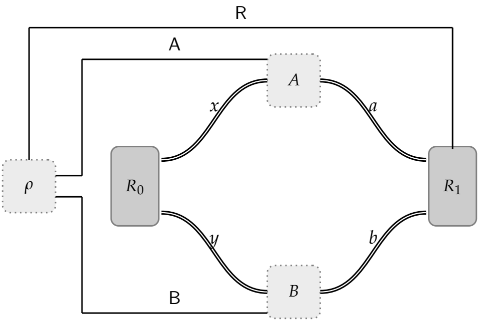
An *extended nonlocal game* is a nonlocal game where the *referee also holds a quantum system* that he measures provided by Alice and Bob.




1. Question and answer sets $(\Sigma_A, \Sigma_B)$ and $(\Gamma_A, \Gamma_B)$.
2. Distribution on question pairs: $\pi : \Sigma_A \times \Sigma_B \to [0, 1]$.
3. A measurement operator $V : \Gamma_A \times \Gamma_B \times \Sigma_A \times \Sigma_B \to \mathrm{Pos}(\mathcal{R})$.

# Extended nonlocal games: Winning and losing probabilities

At the end of the protocol, the referee has:

1. The state at the end of the protocol:

$$\rho_{a,b}^{x,y} \in \mathrm{D}(\mathcal{R}).$$

2. A measurement the referee makes on its part of the state $\rho$:

$$V(a, b|x, y) \in \mathrm{Pos}(\mathcal{R}).$$

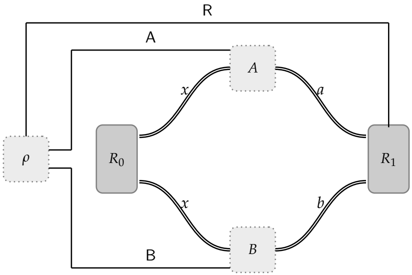
The respective winning and losing probabilities are given by

$$\left\langle V(a, b|x, y), \rho_{a,b}^{x,y} \right\rangle \quad \text{and} \quad \left\langle \mathbb{1} - V(a, b|x, y), \rho_{a,b}^{x,y} \right\rangle.$$

# Monogamy-of-entanglement games

# Monogamy-of-entanglement games

Monogamy-of-entanglement games[¶], are a special type of extended nonlocal game.




1. Same question and answer sets: $\Sigma = \Sigma_A = \Sigma_B$ and $\Gamma = \Gamma_A = \Gamma_B$.
2. Alice and Bob get the same question: $\pi(x, y) = 0$ for $x \neq y$.
3. Referee's measurement operator: $R : \Sigma \times \Gamma \to \mathrm{Pos}(\mathcal{R})$.
4. Winning condition: Iff Alice's output, Bob's output, and the referee's output are all the *equal*.

[¶] [Tomamichel, Fehr, Kaniewski, Wehner, (2013)]

# Standard quantum strategies for monogamy-of-entanglement games

A *standard quantum strategy* consists of a tripartite state $\rho \in \mathrm{D}(\mathcal{R} \otimes \mathcal{A} \otimes \mathcal{B})$ and sets of local measurements for Alice and Bob.

- The winning probability for a monogamy-of-entanglement game using a standard quantum strategy is:

$$\sum_{x \in \Sigma} \pi(x) \sum_{a \in \Gamma} \left\langle R(a|x) \otimes A_a^x \otimes B_a^x, \rho \right\rangle.$$

The standard quantum value of a monogamy-of-entanglement game, $G$, denoted as $\omega^*(G)$, is the maximal winning probability for Alice and Bob over all standard quantum strategies.

# Unentangled strategies for monogamy-of-entanglement games

In an *unentangled strategy*, the state $\rho$ prepared by Alice and Bob is fully separable, that is

$$\{\rho_j^{\mathsf{R}} : j \in \Delta\} \subseteq \mathrm{D}(\mathcal{R}), \quad \{\rho_j^{\mathsf{A}} : j \in \Delta\} \subseteq \mathrm{D}(\mathcal{A}), \quad \{\rho_j^{\mathsf{B}} : j \in \Delta\} \subseteq \mathrm{D}(\mathcal{B}),$$

such that

$$\rho = \sum_{j \in \Delta} p(j) \rho_j^{\mathsf{R}} \otimes \rho_j^{\mathsf{A}} \otimes \rho_j^{\mathsf{B}}.$$

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such that

$$\rho = \sum_{j \in \Delta} p(j) \rho_j^{\mathsf{R}} \otimes \rho_j^{\mathsf{A}} \otimes \rho_j^{\mathsf{B}}.$$

Winning probability for an unentangled strategy is given by:

$$\sum_{x \in \Sigma} \pi(x) \sum_{a \in \Gamma} \left\langle R(a|x) \otimes A_a^x \otimes B_a^x, \rho \right\rangle$$

where $\rho$ is separable.
The *unentangled value*, denoted as $\omega(G)$, is the supremum of the winning probability over all unentangled strategies.

# Unentangled value for monogamy-of-entanglement games

- For $\omega(G)$, we want the *best* Alice and Bob can do.
- Since $\rho$ is separable (no quantum correlations) we can imagine Alice and Bob optimizing functions $f : \Sigma \to \Gamma$ locally.

# Unentangled value for monogamy-of-entanglement games

- For $\omega(G)$, we want the *best* Alice and Bob can do.
- Since $\rho$ is separable (no quantum correlations) we can imagine Alice and Bob optimizing functions $f : \Sigma \to \Gamma$ locally.

Alice and Bob only win when their outputs agree, and we assume that the measurements of the referee are positive semidefinite (from the definition for monogamy-of-entanglement games).

- For any monogamy-of-entanglement game, $G$, the unentangled value is:

$$\omega(G) = \max_{f:\Sigma\to\Gamma} \left\| \sum_{x\in\Sigma} \pi(x) R(f(x)|x) \right\|,$$

where the maximum is over all functions $f : \Sigma \to \Gamma$.

# The BB84 monogamy-of-entanglement game

The BB84 game ($G_{\mathrm{BB84}}$ for short)¶ is defined by:

1. Question and answer sets:

$$\Sigma = \Gamma = \{0, 1\},$$

2. Uniform probability for questions:

$$\pi(0) = \pi(1) = \frac{1}{2}$$

3. Measurements defined by the BB84 bases:

$$\begin{array}{lll} \text{For } x = 0: & R(0|0) = |0\rangle\langle 0|, & R(1|0) = |1\rangle\langle 1| \\ \text{For } x = 1: & R(0|1) = |+\rangle\langle +|, & R(1|1) = |-\rangle\langle -| \end{array}$$

The *unentangled* and *standard quantum* values for $G_{\mathrm{BB84}}$ coincide:

$$\omega(G_{\mathrm{BB84}}) = \omega^*(G_{\mathrm{BB84}}) = \cos^2(\pi/8) \approx 0.8536$$

¶ $G_{\mathrm{BB84}}$ was introduced in [Tomamichel, Fehr, Kaniewski, Wehner, (2013)].

## Demo Time: BB84 game
BB84_GAME.M

# A natural question for monogamy-of-entanglement games

- *Question:* For any monogamy-of-entanglement game, $G$, is it true that the *unentangled* and *standard quantum* values always coincide? In other words is it true that:

$$\omega(G) = \omega^*(G)$$

for all monogamy-of-entanglement games $G$?

# Demo Time: Random monogamy-of-entanglement games

RANDOM_MOE_GAMES.M

# A natural question for monogamy-of-entanglement games

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  for all monogamy-of-entanglement games $G$?

- *Answer:*
  - For certain cases: Yes.
  - In general: No.

$$\omega(G) = \omega^*(G)$$

In general No

# Monogamy-of-entanglement games where $\omega(G) \neq \omega^*(G)$

There exists a monogamy-of-entanglement game, $G$, with $|\Sigma| = 4$ and $|\Gamma| = 3$ such that

$$\omega(G) < \omega^*(G).$$

1. Question and answer sets:

$$\Sigma = \{0, 1, 2, 3\}, \quad \Gamma = \{0, 1, 2\}.$$

2. Uniform probability for questions:

$$\pi(0) = \pi(1) = \pi(2) = \pi(3) = \frac{1}{4}.$$

3. Measurements defined by a mutually unbiased basis[¶]:

$$\{R(0|x), R(1|x), R(2|x)\}.$$

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[¶] $|u_x(a)^* u_{x'}(a)|^2 = 1/|\Gamma|$ for $R(a|x) = u_x(a)u_x(a)^*, R(a|x') = u_{x'}(a)u_{x'}(a)^*$

Demo Time: Mutually unbiased basis game

mub_4_3_game.m

# Monogamy-of-entanglement games where $\omega(G) \neq \omega^*(G)$

- An exhaustive search over all unentangled strategies reveals an optimal unentangled value:

$$\omega(G) = \frac{3 + \sqrt{5}}{8} \approx 0.6545.$$

- Alternatively, a computer search over standard quantum strategies and a heuristic approximation for the upper bound of $\omega^*(G)$ reveals that:

$$2/3 \geq \omega^*(G) \geq 0.6609.$$

This ability to compute upper bounds for extended nonlocal games is obtained from an adaptation of a technique known as the *NPA hierarchy*.

$$\omega(G) = \omega^*(G)$$

For certain classes, Yes.

# Monogamy games that obey $\omega(G) = \omega^*(G)$

Theorem (Johnston, Mittal, R, Watrous)

*For any monogamy-of-entanglement game, $G$, for which $|\Sigma| = 2$:*

$$\omega(G) = \omega^*(G).$$

# Parallel repetition of monogamy-of-entanglement games

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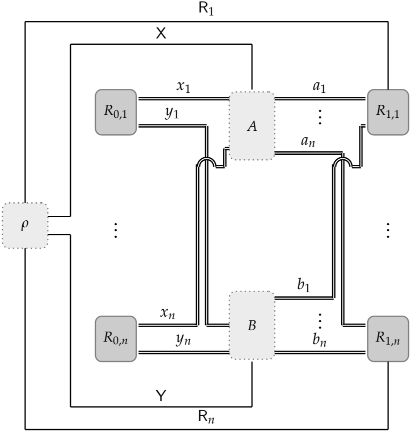
- *Parallel repetition:* Run a monogamy-of-entanglement game, $G$, for $n$ times in parallel, denoted as $G^n$.
- *Strong parallel repetition:* $\omega(G^n) = \omega(G)^n$




*Question:* Do all monogamy-of-entanglement games obey strong parallel repetition?

# Parallel repetition of monogamy-of-entanglement games

- Recall:

$$\omega(G_{\mathrm{BB84}}) = \omega^*(G_{\mathrm{BB84}}) = \cos^2(\pi/8) \approx 0.8536.$$

- $G_{\mathrm{BB84}}$ obeys strong parallel repetition[¶]:

$$\omega^*(G^n_{\mathrm{BB84}}) = \omega^*(G_{\mathrm{BB84}})^n = \left(\cos^2(\pi/8)\right)^n.$$

¶ [Tomamichel, Fehr, Kaniewski, Wehner, (2013)]

# Demo Time: Strong parallel repetition of BB84

bb84_parallel_rep.m

# Upper bounds on strong parallel repetition for monogamy games

Theorem (Tomamichel, Fehr, Kaniewski, Wehner)

*Let $G = (\pi, R)$ be a monogamy game where $\pi$ is uniform over $\Sigma$. It holds that*

$$\omega^*(G^n) \leq \left( \frac{1}{|\Sigma|} + \frac{|\Sigma| - 1}{|\Sigma|} \sqrt{c(G)} \right)^n ,$$

*where $c(G)$ is the "maximal overlap of measurements" of the referee*

$$c(G) = \max_{\substack{x,y \in \Sigma \\ x \neq y}} \max_{a,b \in \Gamma} \left\| \sqrt{R(a|x)} \sqrt{R(b|y)} \right\|^2 .$$

# Strong parallel repetition for certain monogamy games

Theorem (Johnston, Mittal, R, Watrous)

*Let $G = (\pi, R)$ be a monogamy game where $|\Sigma| = 2$, $\pi$ is uniform over $\Sigma$, and $R(a|x)$ are projective operators. It holds that*

$$\omega^*(G^n) = \left(\frac{1}{2} + \frac{1}{2}\sqrt{c(G)}\right)^n .$$

# A key proposition and lemma

**Lemma**

*Let $\Pi_0$ and $\Pi_1$ be nonzero projection operators on $\mathbb{C}^n$. It holds that*

$$\|\Pi_0 + \Pi_1\| = 1 + \|\Pi_0\Pi_1\|.$$

**Proposition**

*Let $G = (\pi, R)$ be a monogamy-of-entanglement game for which $\Sigma = \{0,1\}$, $\pi$ is uniform over $\Sigma$, and $R(a|x)$ is a projection operator for each $x \in \Sigma$ and $a \in \Gamma$. It holds that*

$$\omega(G) = \frac{1}{2} + \frac{1}{2}\max_{a,b\in\Gamma}\Big\| R(a|0)R(b|1) \Big\|.$$

# Proof of proposition

Recall that the unentangled value for any monogamy game $G$ is written as

$$\omega(G) = \max_{f:\Sigma\to\Gamma} \left\| \sum_{x\in\Sigma} \pi(x) R(f(x)|x) \right\|.$$

Assuming the lemma stating $\|\Pi_0 + \Pi_1\| = 1 + \|\Pi_0\Pi_1\|$, we have

$$\omega(G) = \max_{a,b\in\Gamma} \left\| \frac{R(a|0) + R(b|1)}{2} \right\| = \frac{1}{2} + \frac{1}{2} \max_{a,b\in\Gamma} \left\| R(a|0)R(b|1) \right\|.$$

# Proof of theorem

From the proposition that

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$$\omega(G^n) = \left(\frac{1}{2} + \frac{1}{2}\sqrt{c(G)}\right)^n.$$

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Recall that the theorem from [Tomamichel, Fehr, Kaniewski, Wehner, (2013)] gives us

$$\omega^*(G^n) \leq \left(\frac{1}{2} + \frac{1}{2}\sqrt{c(G)}\right)^n,$$

which gives us that $\omega^*(G^n) \leq \omega(G^n)$.

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$$\omega^*(G^n) \leq \left(\frac{1}{2} + \frac{1}{2}\sqrt{c(G)}\right)^n,$$

which gives us that $\omega^*(G^n) \leq \omega(G^n)$. Finally,

$$\omega^*(G^n) \geq \omega(G^n) \geq \left(\frac{1}{2} + \frac{1}{2}\max_{a,b\in\Gamma}\left\|R(a|0)R(b|1)\right\|\right)^n = \left(\frac{1}{2} + \frac{1}{2}\sqrt{c(G)}\right)^n.$$

# Open questions

# Unentangled vs. standard quantum strategies for monogamy-of-entanglement games

| Inputs ($\|\Sigma\|$) | Outputs ($\|\Gamma\|$) | $\omega^*(G) = \omega(G)$ | $\omega^*(G^n) = \omega^*(G)^n$ | $\omega_{ns}(G^n) = \omega_{ns}(G)^n$ |
|---|---|---|---|---|
| 2 | $\|\Gamma\| \geq 1$ | yes | yes¶ | no |
| 3 | $\|\Gamma\| \geq 1$ | ? | ? | no |
| 4 | 3 | no | ? | no |

Question: What about $|\Sigma| = 3$?

- Proof technique fails for $|\Sigma| > 2$.
- Computational search:
  - Generate random monogamy-of-entanglement games where $|\Sigma| = 3$ and $|\Gamma| \geq 2$.
  - $10^8$ random games generates, no counterexamples found.

¶So long as the measurements used by the referee are projective and the probability distribution, $\pi$, from which the questions are asked is uniform.

Thanks!